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Research Article

Blowing Up for the p-Laplacian Parabolic Equation with Logarithmic Nonlinearity

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In this article, we are concerned with a problem for the p-Laplacian parabolic equation with logarithmic nonlinearity; the blow-up result of the solution is proven. This work is completed Boulaaras' work in Math. Methods Appl. Sci., (2020), where the author did not study the blowup of the solution.

1. Introduction

In the current manuscript, we consider the following initial-boundary value problem for a nonlinear p-Laplacian equation:

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + |u|^{p-2}u = |u|^{p-2}u \ln |u|, x \in \Omega, t > 0, \\ u(x,0) = u_0(x), x \in \Omega, \\ u(x,t) = 0, x \in \partial\Omega, t \geq 0, \end{cases} \tag{1}$$

where $\Omega \subset R^n$ is a bounded domain with smooth boundary $\partial\Omega$ and u_0 is the initial data p satisfying

$$\begin{cases} 2 < p < \infty, & \text{if } n \leq p, \\ 2 < p < \dfrac{np}{n-p}, & \text{if } n > p. \end{cases} \tag{2}$$

The terminology of nonlinear polynomials is among the work that researchers have focused on recently. For example, it is found in edge detection and optical elasticity, materials science, engineering, physics, and photonics. In addition, many works and problems in applied sciences have been designed and proposed by means of partial differential equations, including the modeling of some dynamic systems in physics and engineering ([1–13]).

The same is said for the evolutionary partial differential equations associated with $p(x)$-Laplacian (see [8, 14, 15]).

We also note that logarithmic nonlinearity has been concerned by many scientists and researchers, and it has introduced many issues, including the wave equation (see [3, 16–18]).

And for more information on some of the other works to which this term was introduced, we refer the reader to [13, 14, 16–24].

Later on, in [25], the authors by the multiplier method gave the energy decay of the solution of the following problem:

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) - \Delta u_t + |u_t|^{q-1}u_t = |u|^{p-1}u. \tag{3}$$

In addition, the authors in [14] proved the decay rate of solutions (exponential and polynomial) by using the inequality of Nakao for the seminar problem (3).

On the other hand, for the Laplacian parabolic equation with the logarithmic source term in [21], Chen et al. studied the following problem:

$$u_t - \Delta u - \Delta u_t = u \ln u. \tag{4}$$

Then, in [23], the authors proved the global existence, the decay, and the blowup of the solutions of the problem:

$$u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) - \Delta u_t = |u|^{p-2}u \ln |u|, \tag{5}$$

where $p > 2$.

Also, in [14], the authors established the global boundedness and the blowup of the solution of the problem (5) for $1 < p < 2$.

Motivated by the last recent mentioned works, here, we investigated problem (1) with the nonlinear diffusion $\Delta_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and logarithmic nonlinearity $|u|^{p-2}u \ln|u|$ which extends problem in [14]. Our goal is to blow up solutions for problem (1) in order to put some preliminaries. More precisely, we give the blow-up result.

2. Preliminaries

As a starting point, we gave some essential definitions and lemmas.

$$\|u\|_p = \|u\|_{L^p(\Omega)}, \|u\|_{1,p} = \|u\|_{W_0^{1,p}(\Omega)} = \left(\|u\|_p + \|\nabla u\|_p\right)^{1/p}, \tag{6}$$

for $1 < p < \infty$, and we symbolize the positive constants by C and C_i $(i = 1, 2, \cdots)$.

Lemma 1 [7] (logarithmic Sobolev inequality). *Let u be all function $u \in W_0^{1,p}(R^n) \setminus \{0\}$. Then, for $p > 1$, $\mu > 0$,*

$$p\int_{R^n} u^p \ln\left(\frac{|u|}{\|u\|_{L^p(R^n)}}\right) dx \le \mu \int_{R^n} |\nabla u|^p dx - \frac{n}{p} \ln\left(\frac{p\mu e}{n\mathscr{L}_p}\right)\int_{R^n} |u|^p dx, \tag{7}$$

where

$$\mathscr{L}_p = \frac{p}{n}\left(\frac{p-1}{e}\right)^{p-1} \pi^{-p/2}\left[\frac{\Gamma((n/2)+1)}{\Gamma(n((p-1)/p)+1)}\right]^{p/n}. \tag{8}$$

Remark 2. Let $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, and by defining $u(x) = 0$ for $x \in R^n \setminus \Omega$, we can write

$$p\int_{\Omega} u^p \ln\left(\frac{|u|}{\|u\|_{L^p(\Omega)}}\right) dx \le \mu \int_{\Omega} |\nabla u|^p dx - \frac{n}{p} \ln\left(\frac{p\mu e}{n\mathscr{L}_p}\right)\int_{\Omega} |u|^p dx. \tag{9}$$

3. Blowup

In this third section, we gave the proof of blowup of solution of our problem.

Theorem 3. *For any initial data $u_0 \in \mathscr{H}$, the problem (1) has a unique weak solution:*

$$u \in C([0, T]; \mathscr{H}), \tag{10}$$

for some $T > 0$.

First, we introduce the energy functional in the following lemma.

Lemma 4. *Let $u(t)$ be a solution of (1), then $E(t)$ is nonincreasing; that is,*

$$E(t) = \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{p}\int_{\Omega} \ln|u| u^p dx + \frac{p+1}{p^2}\|u\|_p^p \tag{11}$$

satisfies

$$E'(t) = -\|u_t\|_2^2. \tag{12}$$

Proof. Multiplying (1) by u_t and integrating on Ω, we have

$$\begin{aligned} &-\int_{\Omega} \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)u_t dx + \int_{\Omega} |u|^{p-2}u u_t dx + \int_{\Omega} u_t u_t dx \\ &\quad = \int_{\Omega} u^{p-2}u \ln|u| u_t dx, \\ &\frac{d}{dt}\left(\frac{1}{p}\|\nabla u\|_p^p + \frac{1}{p}\|u\|_p^p - \frac{1}{p}\int_{\Omega} \ln|u| u^p dx + \frac{1}{p^2}\|u\|_p^p\right) \\ &\quad = -\|u_t\|^2. \end{aligned} \tag{13}$$

Thus,

$$E'(t) = -\|u_t\|^2. \tag{14}$$

□

To get to our goal of proving the main result, we define the functional

$$H(t) = -E(t) = -\frac{1}{p}\|\nabla u\|_p^p + \frac{1}{p}\int_{\Omega} \ln|u| u^p dx - \frac{p+1}{p^2}\|u\|_p^p. \tag{15}$$

Theorem 5. *Assume that $E(0) < 0$, then the solution of problem (1) blows up in finite time.*

Proof. From (12), we have

$$E(t) \le E(0) \le 0. \tag{16}$$

Hence,

$$H'(t) = -E'(t) = \|u_t\|_2^2 \geq 0,\\ 0 \leq H(0) \leq H(t) \leq \frac{1}{p}\int_\Omega \ln|u|u^p dx. \tag{17}$$

We set

$$\mathscr{K}(t) = H^{1-\alpha} + \frac{\varepsilon}{2}\int_\Omega u^2 dx, \tag{18}$$

where $\varepsilon > 0$ and

$$0 < \alpha < \frac{p-2}{p} < 1. \tag{19}$$

Multiplying (1) by u and the derivative of (18) gives

$$\mathscr{K}' - \alpha H'(t) - \varepsilon\|\nabla u\|_p^p - \varepsilon\|\nabla u\|_p^p + \varepsilon\int_\Omega |u|^p \ln|u| dx. \tag{20}$$

Adding and subtracting $\varepsilon\delta H(t)$ into (20) ($\delta > 0$), we obtain

$$\mathscr{K}' - \alpha H'(t) + \varepsilon\left(\frac{\delta - p}{p}\right)\|\nabla u\|_p^p\\ + \varepsilon\left(\frac{\delta - p}{p} + \frac{1}{p^2}\right)\|u\|_p^p - \varepsilon\left(\frac{\delta - p}{p}\right)\int_\Omega \ln|u|u^p dx + \varepsilon\delta H(t). \tag{21}$$

Applying the logarithmic Sobolev inequality gives

$$\mathscr{K}' - \alpha H'(t) + \varepsilon\delta H(t) + \varepsilon\left(\frac{\delta - p}{p}\right)\left(1 - \frac{\mu}{p}\right)\|\nabla u\|_p^p + \varepsilon\left(\frac{\delta - p}{p}\right)\\ \cdot\left[1 + \frac{\delta}{p(\delta - p)} - \ln\|u\|_p + \left(\frac{n}{p^2}\ln\left(\frac{p\mu e}{n\mathscr{L}_p}\right)\right)\right]\|u\|_p^p. \tag{22}$$

Setting $\mu = p/2$ and taking $\delta > p$ give

$$\left[1 + \frac{\delta}{p(\delta - p)} - \ln\|u\|_p + \left(\frac{n}{p^2}\ln\left(\frac{p^2 e}{2n\mathscr{L}_p}\right)\right)\right] > 0, \tag{23}$$

since

$$\|u\|_p > e^{\left(1 + \frac{\delta}{p(\delta - p)}\right)}\left(\frac{p^2 e}{2n\mathscr{L}_p}\right)^{n/p^2}. \tag{24}$$

Consequently, for some $\beta > 0$, inequality (25) gives

$$\mathscr{K}'(t) \geq \beta\left\{H(t) + \|u\|_p^p + \|\nabla u\|_p^p\right\}, \tag{25}$$

$$\mathscr{K}(t) \geq \mathscr{K}(0) > 0, \quad t > 0. \tag{26}$$

Next, by (18), we have

$$\mathscr{K}(t) = H^{1-\alpha} + \frac{\varepsilon}{2}\int_\Omega u^2 dx \leq H^{1-\alpha} + \varepsilon C\|u\|_p^2\\ \leq H^{1-\alpha} + \varepsilon C\left(\|u\|_p^p\right)^{2/p}. \tag{27}$$

Therefore,

$$\mathscr{K}^{1/1-\alpha}(t) \leq H^{1-\alpha} + \varepsilon C\left(\|u\|_p^p\right)^{2/p(1-\alpha)}, \tag{28}$$

where $0 < 2/p(1-\alpha) < 1$,

$$\left(\|u\|_p^2\right)^{2/p(1-\alpha)} \leq C\left(\left(\|u\|_p^p\right)^p + H(t)\right). \tag{29}$$

Hence,

$$\mathscr{K}^{1/1-\alpha}(t) \leq C_1\left[H(t) + \|u\|_p^p\right] \leq C_1\left[H(t) + \|\nabla u\|_p^p + \|u\|_p^p\right]. \tag{30}$$

According to (25) and (30), we get

$$\mathscr{K}'(t) \geq \lambda\mathscr{K}^{1/1-\alpha}(t), \tag{31}$$

where $\lambda = C_1/\beta > 0$, depending only on β and C_1.

Finally, by integrating (31), we obtain

$$\mathscr{K}^{\alpha/1-\alpha}(t) \geq \frac{1}{\mathscr{K}^{-\alpha/(1-\alpha)}(0) - \lambda(\alpha/(1-\alpha))t}. \tag{32}$$

Hence, $\mathscr{K}(t)$ blows up in time:

$$T \leq T^* = \frac{1-\alpha}{\lambda\alpha\mathscr{K}^{\alpha/(1-\alpha)}(0)}. \tag{33}$$

As a result, the proof is completed. □

Data Availability

No data were used to support the study.

Conflicts of Interest

The author declares that he has no conflicts of interest.

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