Journal of Advances in Mathematics and Computer Science

Volume 37, Issue 11, Page 31-69, 2022; Article no.JAMCS.93690

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

The Solution of the Invariant Subspace Problem. Complex Hilbert Space. External Countable Dimensional Linear spaces Over Field ${}^{*}\mathbb{R}_{c}^{\#}$. Part II.

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/JAMCS/2022/v37i111721

Open Peer Review History:
This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/93690

Received: 20/09/2022
Accepted: 26/11/2022
Published: 06/12/2022

Original Research Article

Abstract

We present a new approach to the invariant subspace problem for complex Hilbert spaces.This approach based on nonconservative Extension of the Model Theoretical NSA. Our main result will be that: if T is a bounded linear operator on an infinite-dimensional complex separable Hilbert space H,it follow that T has a non-trivial closed invariant subspace.

Keywords: Set theory ZFC; Nonconservative extension of ZFC; Internal set theory IST; External set theory HST; A. Robinson model theoretical NSA; Bivalent gyper infinitary logic; Modus ponens rule; Logic with restricted modus ponens rule; internal non-Archimedean field; Invariant subspce problem.

2010 Mathematics Subject Classification: 03E30, 03E65, 03E70, 03H05, 03H15, 05D05, 03C75.

1 Introduction

The incompleteness of set theory ZFC leads one to look for natural extensions of ZFC in which one can prove statements independent of ZFC which appear to be "true".

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One approach has been to add large cardinal axioms. Or, one can investigate second-order expansions like Kelley-Morse class theory, KM or Tarski-Grothendieck set theory TG [1]-[3].It is a non-conservative extension of ZFC and is obtaineed from other axiomatic set theories by the inclusion of Tarski's axiom which implies the existence of inaccessible cardinals. Non-conservative extension of ZFC based on an generalized quantifiers considered in [4]. In this paper we look at a set theory $\mathbf{NC}^{\#}_{\infty\#}$,based on bivalent hyper infinitary logic ${}^{2}L^{\#}_{\infty\#}$ with restricted Modus Ponens Rule [5]-[8].Set theory $\mathbf{NC}^{\#}_{\infty\#}$contains Aczel's anti-foundation axiom [9].

Non-conservative extension based on set theory $\mathbf{NC}^{\#}_{\infty\#}$ of the model theoretical nonstandard analysis [10]-[11] also is considered [8].

1.1 The invariant subspace problem. Positive classical results

The problem, in a general form, is stated as follows. The Invariant Subspace Problem: If T is a bounded linear operator on an infinite-dimensional separable Hilbert space H, does it follow that T has a non-trivial closed invariant subspace?.

The Invariant Subspace Problem (as it stands today). If T is a bounded linear operator on an infinite-dimensional separable Hilbert space H, does it follow that T has a non-trivial closed invariant subspace?.

Sometime during the 1930s John von Neumann proved that compact operators have non-trivial invariant subspaces, but did not publish it. The proof was rediscovered and finally published by N. Aronszajn and K. T. Smith [12] in 1954.

Theorem 1.1. (Von Neumann). Every compact operator on H has a non-trivial invariant subspace.
In 1966 Bernstein and Robinson [13] extended the result to the slightly larger class of polynomially compact operators, see also [14].

Definition 1.1. A linear operator T on a Banach space is said to be polynomially compact if there is a non-zero polynomial $p(t) \in \mathbb{C}[t]$ such that $p(T)$ is compact.

An nonclassical aspect of Bernstein and Robinson's proof is that it used the relatively new techniques of non-standard analysis, which builds up the foundations of analysis based on a rigorous definition of infinitesimal numbers. Shortly after, the proof was translated into standard analysis by Halmos [15].

The next major generalization was achieved by Arveson and Feldman [16] in1968.

Definition 1.2. For a bounded linear operator T on X, the uniformly closed algebra generated by T, denoted by $\mathbf{A}(T)$, is defined to be the subspace $[\{I, T, T^2, ...\}]$ of $\mathbf{B}(X)$.Alternatively, $\mathbf{A}(T)$ is the smallest closed subspace of $\mathbf{B}(X)$ containing T and I which is closed under function composition.

If T is a bounded operator, then $\mathbf{A}(T)$ can be thought of as the closure of the set of polynomial combinations of T, or the set of all operators which can be norm approximated by polynomial combinations of T.

Theorem 1.2.(Arveson and Feldman [16]). If $T : H \to H$ is a bounded quasinilpotent operator such that $\mathbf{A}(T)$ contains a non-zero compact operator, then T has a non-trivial invariant subspace.

While the techniques of von Neumann and subsequent generalizations yielded many interesting and surprising theorems during the 1950s and 60s, their effectiveness was reaching its limit by the 70s.

1.2 The invariant subspace problem. Positive nonclassical results

A new approach to invariant subspace problem for complex Hilbert spaces originally has been presented in author papers [17]-[18].This approach based on non-conservative Extension of the Model Theoretical NSA. The main result will be that: if T is a bounded linear operator on an infinite-dimensional complex separable Hilbert space H, it follow that T has a non-trivial closed invariant subspace [18]. This approach based on analysis on external non-Archimedean field $\mathbb{R}_c^{\#}$, see [17]-[19].

1.3 The invariant subspace problem of oprator algebras. Positive non-classical results

Let X be a Banach space; let (X) be the lattice with the operations of intersection $\wedge$ and of taking the closed linear hull V of all of its closed subspaces; and let $B(X)$ be the algebra of all bounded linear operators in X. A subspace $\Im \in (X)$ is said to be invariant (briefly IS) with respect to a family of operators $\Re \subset B(X)$, or $\Re$-invariant, if $Ax \in \Im, \forall x \in \Im \ A \in \Re$. The collection of all $\Re$-invariant subspaces is denoted by $\mathbf{lat}\,(\Re)$; obviously, $\mathbf{lat}\,(\Re)$ is a complete sublattice in (X).

In the case when $\mathbf{dim}\,(X) < \infty$, the fundamental problem of the existence of an IS is solved by Burnside's theorem: $\mathbf{lat}\,(\Re) = \{0, X\} \iff (\Re) = B(X)$ where $(\Re)$ is the algebra generated by $\Re$. The question of the validity of an analogous statement with the replacement of $(\Re)$ by its weak closure for a Banach space is called Burnside's problem in X; presently, Burnside's problem is open from Burnside time until nowadays for all spaces in which the IS problem is not solved by using canonical approuch.

Theorem 1.3. Let H be infinite-dimensional complex separable Hilbert space. Then $\mathbf{lat}\,(\Re) = \{0, H\} \iff (\Re) = B(H)$.

2 External Hyper finite Dimensional Linear Spaces. Subspaces, Direct Summ and Factor Spaces

2.1 Basic results and definitions

Definition 2.1.[18]. A vector space over a field ${}^*\mathbb{R}_c^{\#}$ is a set V together with two operations that satisfy the eight axioms listed below.

The first operation, called vector addition or simply addition $+ : V \times V \to V$, takes any two vectors $\mathbf{x}$ and $\mathbf{y}$ and assigns to them a third vector which is commonly written as $\mathbf{x} + \mathbf{y}$, and called the sum of these two vectors.

The second operation, called scalar multiplication $\times : F \times V \to V$ff0c takes any scalar a and any vector $\mathbf{v}$ and gives another vector $a \times \mathbf{x}$.

Axioms:

(1) $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$;
(2) $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$;
(3) There exists $\mathbf{0} \in V$ such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for every $\mathbf{x} \in V$;
(4) For every $\mathbf{x} \in V$ there exists $\mathbf{y} \in V$ such that $\mathbf{x} + \mathbf{y} = \mathbf{0}$;
(5) $1 \times \mathbf{x} = \mathbf{x}$ for every $\mathbf{x} \in V$;
(6) $\alpha(\beta\mathbf{x}) = (\alpha\beta)\,\mathbf{x}$ for every $\mathbf{x} \in V$ and every $\alpha,\beta,\gamma \in {}^*\mathbb{R}_c^{\#}$;

(7) $(\alpha+\beta)\times\mathbf{x}=\alpha\times\mathbf{x}+\beta\times\mathbf{x}$ for every $\mathbf{x}\in V$ and every $\alpha,\beta\in{}^{*}\mathbb{R}_c^{\#}$;
(8) $\alpha\times(\mathbf{x}+\mathbf{y})=\alpha\times\mathbf{x}+\alpha\times\mathbf{y}$ for every $\mathbf{x},\mathbf{y}\in V$ and every $\alpha\in{}^{*}\mathbb{R}_c^{\#}$.

Axioms (1)-(8) have a number of implications:

Theorem 2.1.[18]. The zero vector $\mathbf{0}$ in a linear space V is unique.

Proof. The existence of at least one zero vector is asserted in axiom (3). Suppose there are two zero vectors $\mathbf{0}_1$ and $\mathbf{0}_2$ in the space V. Setting $\mathbf{x}=\mathbf{0}_1,\mathbf{0}=\mathbf{0}_2$ in axiom (3), we obtain $\mathbf{0}_1+\mathbf{0}_2=\mathbf{0}_1$.Setting $\mathbf{x}=\mathbf{0}_2$,$\mathbf{0}=\mathbf{0}_1$ in axiom (3),weobtain $\mathbf{0}_1+\mathbf{0}_2=\mathbf{0}_2$.

Comparing the first of these relations with the second and using axiom (1), we find that $\mathbf{0}_1=\mathbf{0}_2$.

Theorem 2.2.[18]. Every element in a linear space has a unique vnegative.

Proof. The existence of at least one negative element is asserted in axiom (4).

Suppose an element $\mathbf{x}\in V$ has two negatives y_1 and y_2. Adding y_2 to both sides of the equation $\mathbf{x}+\mathbf{y}_1=\mathbf{0}$ and using axioms (1)-(3), we get$\mathbf{y}_2+(\mathbf{x}+\mathbf{y}_1)=(\mathbf{y}_2+\mathbf{x})+\mathbf{y}_1=\mathbf{0}+\mathbf{y}_1=\mathbf{y}_1,\mathbf{y}_2+(\mathbf{x}+\mathbf{y}_1)=\mathbf{y}_2+\mathbf{0}=\mathbf{y}_2$,whence $\mathbf{y}_1=\mathbf{y}_2$.

Theorem 2.3.[18]. The relation $0\times\mathbf{x}=\mathbf{0}$ holds for every $\mathbf{x}\in V$.

Theorem 2.4.[18]. For any $\mathbf{x}\in V$ the element $\mathbf{y}=(-1)\times\mathbf{x}$ is a negative of $\mathbf{x}$.

Definition 2.2.[18].Let $\mathbf{x}_1,\mathbf{x}_2,...,\mathbf{x}_k,k\in\mathbb{N}^{\#}$ be vectors of the linear space V over a field ${}^{*}\mathbb{R}_c^{\#}$, and let $\alpha_1,\alpha_2,...,\alpha_k$ be numbers from ${}^{*}\mathbb{R}_c^{\#}$. Then the vector

$$\mathbf{y}=Ext\text{-}{}_{i=1}^{k}\alpha_i\mathbf{x}_i \tag{2.1}$$

is called a linear combination of the vectors $\mathbf{x}_1,\mathbf{x}_2,...,\mathbf{x}_k$, and the numbers $\alpha_1,\alpha_2,...,\alpha_k$ are called the coefficients of the linear combination.If $\alpha_i=0,1\leq i\leq k$, then $\mathbf{443}=0$ by

Theorem 2.5.[18].However, there may exist a linear combination of the vectors $\mathbf{x}_1,\mathbf{x}_2,...,\mathbf{x}_k$ which equals the zero vector, even though its coefficients are not all zero.

In this case, the vectors $\mathbf{x}_1,\mathbf{x}_2,...,\mathbf{x}_k$ are called linearly dependent. In other words, the vectors $\mathbf{x}_1,\mathbf{x}_2,...,\mathbf{x}_k$ are said to be linearly dependent if there exist numbers $\alpha_1,\alpha_2,...,\alpha_k\in{}^{*}\mathbb{R}_c^{\#}$, not all equal to zero, such that

$$Ext\text{-}{}_{i=1}^{k}\alpha_i\mathbf{x}_i=0. \tag{2.2}$$

If (2.2) holds if and only if $\alpha_i\equiv 0,1\leq i\leq k$, the vectors $\mathbf{x}_1,\mathbf{x}_2,...,\mathbf{x}_k$ are said to be linearly dependent over ${}^{*}\mathbb{R}_c^{\#}$.

Next we note two simple properties of systems of vectors, both involving the notion of linear dependence.

Theorem 2.6.[18]. If some of the vectors $\mathbf{x}_1,\mathbf{x}_2,...,\mathbf{x}_k$ are linearly dependent, then the whole system $\mathbf{x}_1,\mathbf{x}_2,...,\mathbf{x}_k$ is also linearly dependent.

Proof. Without loss of generality, we can assume that the vectors $\mathbf{x}_1,\mathbf{x}_2,...,\mathbf{x}_j,j<k$ are linearly dependent.

Thus there is a relation

$$Ext\text{-}_{i=1}^{j}\alpha_i\mathbf{x}_i = 0, \tag{2.3}$$

where at least one of the constants $\alpha_1, \alpha_2, ..., \alpha_j$ is different from zero.

By Theorem 2.3 and axiom (3), we have

$$Ext\text{-}_{i=1}^{j}\alpha_i\mathbf{x}_i + Ext\text{-}_{i=j+1}^{k}0 \times \mathbf{x}_i = 0. \tag{2.4}$$

But then the vectors $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k$ are also linearly dependent, since at least one of the constants $\alpha_1, \alpha_2, ..., \alpha_j, 0, ..., 0$ is different from zero. |

Theorem 2.7.[18]. The vectors $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k$ are linearly dependent if and only if one of the vectors can be expressed as a linear combination of the others.

Proof A similar statement has already been encountered; in fact, it was proved for columns of hyperreal numbers in [18]**.** Inspecting the proof given there, we see that it is based only on the possibility of performing on columns the operations of addition and multiplication by hyperreal numbers. Hence the proof can be carried through for the elements of any linear space, i.e., this theorem is valid for any linear space.

Definition 2.3.[18]. A hyperfinite system of linearly independent vectors $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n$, $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$ in a linear space V over a field ${}^*\mathbb{R}_c^{\#}$ is called a basis for V if, given any $\mathbf{x} \in V$, there exists an expansion

$$\mathbf{x} = Ext\text{-}_{i=1}^{n}\zeta_i\mathbf{e}_i, \tag{2.5}$$

where $\zeta_i \in {}^*\mathbb{R}_c^{\#}$, $1 \leq i \leq n$.

It is easy to see that under these conditions the coefficients in the expansion (2.3) are uniquely determined. In fact, if we can write two expansions:

$$\begin{aligned}\mathbf{x} = Ext\text{-}_{i=1}^{n}\zeta_i\mathbf{e}_i,\\ \mathbf{x} = Ext\text{-}_{i=1}^{n}\eta_i\mathbf{e}_i,\end{aligned} \tag{2.6}$$

for a vector $\mathbf{x}$, then, subtracting them term by term, we obtain the relation

$$Ext\text{-}_{i=1}^{n}\left(\zeta_i - \eta_i\right)\mathbf{e}_i = 0 \tag{2.7}$$

from which, by the assumption that the vectors $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n$ are linearly independent, we obtain that

$$\zeta_i = \eta_i, , 1 \leq i \leq n. \tag{2.8}$$

Definition 2.4.[18].The uniquely defined numbers $\zeta_i \in {}^*\mathbb{R}_c^{\#}$, $1 \leq i \leq n$, are called the components of the vector $\mathbf{x}$ with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n$.

Example 2.1. An example of a basis in the space $V_n, n \in \mathbb{N}^{\#}/\mathbb{N}$ is the hyperfinite system of vectors $e_1 = (1, 0, ...)\,, e_2 = (0, 1, ...)\,, ..., e_n = (0, 0, ...1)\,.$ Indeed it is obvious that the relation

$$\mathbf{x} = Ext\text{-}_{i=1}^{n}\zeta_i e_i \tag{2.9}$$

holds for every vector

$$\mathbf{x} = \left(\xi_1, \xi_2, ..., \xi_n\right). \tag{2.10}$$

This fact, together with the linear independence of the vectors $e_i, 1 \leq n$ already proved, shows that these vectors form a basis in the space V_n. In particular, we see that the hyperreal numbers $\xi_i, 1 \leq i \leq n$ are just the components of the vector $\mathbf{x}$ with respect to the basis $e_i, 1 \leq i \leq n$.

Theorem 2.8.[18].When two vectors of a linear space V_n are added, their components (with respect to any basis) are added. When a vector is multiplied by a number $\lambda \in {}^*\mathbb{R}_c^\#$, all its components are multiplied by λ.

Proof. Let

$$\mathbf{x} = Ext\text{-}_{i=1}^{n}\zeta_i\mathbf{e}_i, \mathbf{y} = Ext\text{-}_{i=1}^{n}\eta_i\mathbf{e}_i. \tag{2.11}$$

Then

$$\mathbf{x} + \mathbf{y} = Ext\text{-}_{i=1}^{n}\left(\zeta_i + \eta_i\right)\mathbf{e}_i, \lambda\mathbf{x} = Ext\text{-}_{i=1}^{n}\lambda\zeta_i\mathbf{e}_i \tag{2.12}$$

by the axioms.

Definition 2.4.[18]. If in a linear space V we can find $n \in \mathbb{N}^\#$ linearly independent vectors while every $n+1$ vectors of the space are linearly dependent, then the number $n \in \mathbb{N}^\#/\mathbb{N}$ is called the dimension of the space V and denoted by $\dim_{{}^*\mathbb{R}_c^\#}(V)$; the space V itself is called n -dimensional and denoted V_n. A linear space in which we can find an hyperfinite number of linearly independent vectors also is called hyperfinite-dimensional.

Theorem 2.9.[18].In a space V of dimension $n \in \mathbb{N}^\#$ there exists a basis consisting of n vectors. Moreover, any set of n linearly independent vectors of the space V is a basis for the space.

Proof. Let $\mathbf{e}_i, 1 \leq i \leq n$ be a hyperfinite system of n linearly independent vectors of the given n-dimensional space V.If $\mathbf{x}$ is any vector of the space, then the set of $n+1$ vectors $\mathbf{x}, \mathbf{e}_i, 1 \leq i \leq n$ is linearly dependent, i.e., there exists a relation of the form

$$\alpha_0\mathbf{x} + Ext\text{-}_{i=1}^{n}\alpha_i\mathbf{e}_i = 0, \tag{2.13}$$

where at least one of the coefficients $\alpha_0, \alpha_i, 1 \leq i \leq n$ is different from zero. Clearly α_0 is different from zero, since otherwise the vectors $\mathbf{e}_i, 1 \leq i \leq n$ would be linearly dependent, contrary to hypothesis. Thus, in the usual way, i.e., by dividing (2.11) by α_0 and transposing all the other terms to the other side, we find that $\mathbf{x}$ can be expressed as a linear combination of the vectors $\mathbf{e}_i, 1 \leq i \leq n$. Since $\mathbf{x}$ is an arbitrary vector of the space V, we have shown that the vectors $\mathbf{e}_i, 1 \leq i \leq n$ form a basis for the space.

The preceding theorem has the following converse.

Theorem 2.10.[18].If there is a basis in the space V, then the dimension of V equals the number of basis vectors.

Proof. Let the vectors $\mathbf{e}_i, 1 \leq i \leq n$ be a basis for V. By the definition of a basis, the vectors $\mathbf{e}_i, 1 \leq i \leq n$ are linearly independent; thus we already have n linearly independent vectors. We now show that any $n+1$ vectors of the space V are linearly dependent. Suppose we are given $n+1$ vectors of the space V :

$$x_1 = Ext\text{-}_{i=1}^{n}\xi_i^{(1)}\mathbf{e}_i, x_2 = Ext\text{-}_{i=1}^{n}\xi_i^{(2)}\mathbf{e}_i, \cdots\cdots\cdots\cdots\cdots\cdots x_{n+1} = Ext\text{-}_{i=1}^{n}\xi_i^{(n+1)}\mathbf{e}_i \tag{2.14}$$

Writing the components of each of these vectors as a column of numbers, we form the matrix

$$A = \begin{Vmatrix} \xi_1^{(1)} & \xi_1^{(2)} & \cdots & \xi_1^{(n+1)} \\ \xi_2^{(1)} & \xi_2^{(2)} & \cdots & \xi_2^{(n+1)} \\ \cdot & \cdot & \cdots & \cdot \\ \xi_n^{(1)} & \xi_n^{(2)} & \cdots & \xi_n^{(n+1)} \end{Vmatrix} \tag{2.15}$$

with n rows and $n+1$ columns. The basis minor of the matrix A is of order $r < n$.

If $r = 0$, the linear dependence is obvious. Let $r > 0$. After specifying the r basis columns, we can still find at least one column which is not one of the basis columns.

But then, according to the basis minor theorem, this column is a linear combination of the basis columns. Thus the corresponding vector of the space V is a linear combination of some other vectors among the given $x_1, x_2, ..., x_{n+1}$.

But in this case, according to Theorem 2.6, the vectors $x_1, x_2, ..., x_{n+1}$ are linearly dependent.

Definition 2.5.[18]. A Complex linear space $V = V\left[{}^*\mathbb{C}_c^{\#}\right]$ that is a linear space over field ${}^*\mathbb{C}_c^{\#} = {}^*\mathbb{R}_c^{\#} + i\ {}^*\mathbb{R}_c^{\#}$.

Note that $V\left[{}^*\mathbb{C}_c^{\#}\right]$ is obviously a real space as well, since the domain of the external complex numbers ${}^*\mathbb{C}_c^{\#}$ contains the domain of hyperreal numbers ${}^*\mathbb{R}_c^{\#}$. However, thedimension $\dim_{{}^*\mathbb{C}_c^{\#}}(V)$ of V as a complex space does not coincide with dimension$\dim_{{}^*\mathbb{R}_c^{\#}}(V)$ of $V\left[{}^*\mathbb{C}_c^{\#}\right]$ as a real space. In fact, if the vectors $\mathbf{e}_i, 1 \leq i \leq n$ are linearly independent in V regarded as a complex space, then the vectors $\mathbf{e}_i, i\mathbf{e}_i, 1 \leq i \leq n$, are linearly independent in V regarded as a real space. Hence the dimension of V regarded as a real space is twice as large as that of V regarded as a complex space.

2.2 Subspaces

Suppose that a set L of elements of a linear space V over field ${}^*\mathbb{R}_c^{\#}$has the following properties:

(a) If $\mathbf{x} \in L, 443 \in L$, then $\mathbf{x} + \mathbf{443}$;
(b) Ifx$\in L$ and $\lambda \in {}^*\mathbb{R}_c^{\#}$ then $\lambda\mathbf{x} \in L$.

Thus L is a set of elements with linear operations defined on them.

We now show that this set is also a linear space. To do so, we must verify that the set L with the operations (a) and (b) satisfies the axioms (1), (2) and (5)-(8) are satisfied, since they hold quite generally for all elements of the space V.

It remains to verify axioms (3) and (4). Let $\mathbf{x}$ be any element of L. Then, by hypothesis, $\lambda\mathbf{x} \in L$ for every $\lambda \in {}^*\mathbb{R}_c^{\#}$. First we choose $\lambda = 0$. Then, since $0 \times x = 0$, the zero vector belongs to the set L, i.e., axiom (3) is satisfied. Next we choose $\lambda = -1$.

Then, by

Theorem 2.4, $(-1) \times x$ is the negative of the element x.

Thus, if an element x belongs to the set L, so does the negative of x. This means that axiom (4) is also satisfied, so that L is a linear space, as asserted.

Definition 2.6.[18]. Every set $L \subset V$ with properties (a) and (b) is called a linear subspace (or simply a subspace) of the space V.

Definition 2.7.[18]. Let L_1 and L_2 be two subspaces of the same linear space V.

Then the set of all vectors $\mathbf{x} \in V$ belonging to both L_1 and L_2 forms a subspace called the intersection of the subspaces L_1 and L_2. The set of all vectors of the form $\mathbf{y} + \mathbf{z}$ where **443** $\in L_1, \mathbf{z} \in L_2$ forms a subspace, denoted by $L_1 + L_2$ and called the sum of the subspaces L_1 and L_2.

We now consider some properties of subspaces which are related to the definitions above. First of all, we note that every linear relation which connects the vectors $\mathbf{x}, \mathbf{y}, ..., \mathbf{z}$ in a subspace L is also valid in the whole space V, and conversely.

In particular, the fact that the vectors $\mathbf{x}, \mathbf{y}, ..., \mathbf{z} \in L$ are linearly dependent holds true simultaneously in the subspace L and in the space V. For example, if every set of $n + 1$ vectors is linearly dependent in the space V, then this fact is true a fortiori in the subspace L. It follows that the dimension of any subspace L of an n-dimensional space V does not exceed the number n, According to Theorem 2.9, in any subspace $L \subset V$ there exists a basis with the same number of vectors as the dimension of L.

Of course, if a basis $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n$ is chosen in V, then in the general case we cannot choose the basis vectors of the subspace L from the vectors $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n$, because none of these vectors may belong to L. However, it can be asserted that if a basis $\mathbf{f}_1, \mathbf{f}_2, ..., \mathbf{f}_n$ is chosen in the subspace L {which, to be explicit, is assumed to have dimension $l < n$), then additional vectors $\mathbf{f}_{l+1}, ..., \mathbf{f}_n$ can always be chosen in the whole space V such that the system $\mathbf{f}_1, \mathbf{f}_2, ..., \mathbf{f}_l, ..., \mathbf{f}_n$ is a basis for all of V.

To prove this, we argue as follows: In the space V there are vectors which cannot be expressed as linear combinations of $\mathbf{f}_1, ..., \mathbf{f}_l$.

Indeed, if there were no such vectors, then the vectors $\mathbf{f}_1, \mathbf{f}_2, ..., \mathbf{f}_l$, which are linearly independent by hypothesis, would constitute a basis for the space V, and then by Theorem 2.9 the dimension of V would be l rather than n.Let $\mathbf{f}_{l+1}$ be any of the vectors that cannot be expressed as a linear combination of $\mathbf{f}_1, \mathbf{f}_2, ..., \mathbf{f}_l$.
Then the System $\mathbf{f}_1, \mathbf{f}_2, ..., \mathbf{f}_l, \mathbf{f}_{l+1}$ is linearly independent. In fact, suppose there were a relation of the form

$$Ext\text{-}\sum_{i=1}^{l+1} \alpha_i \mathbf{f}_i = 0. \tag{2.16}$$

Then if $\alpha_{l+1} \neq 0$, the vecto $\mathbf{f}_{l+1}$ could be expressed as a linear combination of $\mathbf{f}_1, ..., \mathbf{f}_l$ while if $\alpha_{l+1} = 0$, the vectors $\mathbf{f}_1, \mathbf{f}_2, ..., \mathbf{f}_l$ would be linearly dependent. But both these results contradict the construction. If now every vector of the space V can be expressed as a linear combination of $\mathbf{f}_1, ..., \mathbf{f}_l, \mathbf{f}_{l+1}$, then the system $\mathbf{f}_1, \mathbf{f}_2, ..., \mathbf{f}_l, \mathbf{f}_{l+1}$ forms a basis for V with $l + 1 = n$, which concludes our construction. If $l + 1 < n$, then there is a vector $\mathbf{f}_{l+2}$ which cannot be expressed as a linear combination $\mathbf{f}_1, \mathbf{f}_2, ..., \mathbf{f}_l, \mathbf{f}_{l+1}$, and hence we can continue the construction by hyper infinite induction principle [8].

Eventually, after $n - l$ steps, we obtain a basis for the space V.

Definition 2.8.[18]. We say that the vectors $g_1, ..., g_k$ are linearly independent over the subspace $L \subset V$ if the relation

$$Ext\text{-}\sum_{i=1}^{k} \alpha_i \mathbf{g}_i \in L \tag{2.17}$$

implies $\alpha_i = 0, 1 \leq i \leq k$.

If L is the subspace consisting of the zero vector alone, then linear independence over L means ordinary linear independence. Linear dependence of the vectors $\mathbf{g}_1, ..., \mathbf{g}_k$ over the subspace L means that there exists a linear combination $Ext\text{-}\sum_{i=1}^{k} \alpha_i \mathbf{g}_i$ belonging to L, where at least one of the coefficients $\alpha_i, 1 \leq i \leq k$ is nonzero.

Definition 2.9.[18]. The largest possible number of vectors of the space V which are linearly independent over the subspace $L \subset V$ is called the dimension of V over L.If the vectors $\mathbf{g}_1, ..., \mathbf{g}_k$ are linearly independent over the space $L \subset V$ and if the vectors $f_1..., f_l$ are linearly independent in the subspace L, then the vectors $\mathbf{g}_1, ..., \mathbf{g}_k$, $\mathbf{f}_1..., \mathbf{f}_l$ are linearly independent in the whole space V. In fact, if there were a relation of the form

$$Ext\text{-}\sum_{i=1}^{l} \alpha_i \mathbf{f}_i + Ext\text{-}\sum_{i=1}^{k} \beta_i \mathbf{g}_i = 0, \tag{2.18}$$

or equivalently

$$Ext\text{-}\sum_{i=1}^{k} \beta_i \mathbf{g}_i = -\left(Ext\text{-}\sum_{i=1}^{l} \alpha_i \mathbf{f}_i\right) \in L \tag{2.19}$$

then $\beta_i = 0, 1 \leq i \leq k$,by the assumed linear independence of the vectors $\mathbf{g}_1, ..., \mathbf{g}_k$ over L. It follows that $\alpha_i = 0, 1 \leq i \leq k$, by the linear independence of the vectors $\mathbf{f}_1..., \mathbf{f}_l$.

Remark 2.1. The vectors $\mathbf{f}_{l+1}..., \mathbf{f}_n$ constructed above are linearly independent over the subspace L.In fact, if there were a relation of the form

$$Ext\text{-}\sum_{i=l+1}^{n} \alpha_i \mathbf{f}_i = Ext\text{-}\sum_{i=1}^{l} \alpha_i \mathbf{f}_i \tag{2.20}$$

with at least one of the numbers $\alpha_{l+1}, ..., \alpha_n$ not equal to zero, then the vectors $\mathbf{f}_1, ..., \mathbf{f}_n$ would be linearly dependent, contrary to the construction. Hence the dimension of the space V over L is no less than $n - l$. On the other hand, this dimension cannot be greater than $n - l$, since if $n - l + 1$ vectors $\mathbf{h}_1..., \mathbf{h}_{n-l+1}$ say, were linearly independent over L, then the vectors $\mathbf{h}_1..., \mathbf{h}_{n-l+1}, \mathbf{f}_1, ..., \mathbf{f}_l$ of which there are more than n, would be linearly independent in V. Therefore the dimension of V over L is precisely $n - l$.

2.3 The hyperfinite direct sum

Definition 2.10.[18].We say that a linear space L is the hyperfinite direct sum of given subspaces $L_1, ..., L_m \subset L, m \in \mathbb{N}^{\#} \backslash \mathbb{N}$ if: (a) For every $\mathbf{x} \in L$ there exists an expansion

$$\mathbf{x} = Ext\text{-}\sum_{i=1}^{m} x_i, \tag{2.21}$$

where $x_1 \in L_1, .., x_m \in L_m$;

(b) This expansion is unique, i.e., if

$$\mathbf{x} = Ext\text{-}\sum_{i=1}^{m} x_i = Ext\text{-}\sum_{i=1}^{m} y_i, \tag{2.22}$$

where $x_j \in L_j, y_j \in L_j, 1 \leq j \leq m$, then $z_i = 0, 1 \leq j \leq m$.

However, the validity of condition (b) is a consequence of the following simpler condition: (b′) If

$$Ext\text{-}\sum_{i=1}^{m} z_i = 0 \tag{2.23}$$

where $z_i \in L_i, 1 \leq i \leq m$, then $z_i = 0, 1 \leq i \leq m$.

In fact, given two expansions $\mathbf{x} = Ext\text{-}_{i=1}^{m} x_i$, $\mathbf{x} = Ext\text{-}_{i=1}^{m} y_i$suppose (b′) holds. Then subtracting the second expansion from the first, we get $\mathbf{0} = Ext\text{-}_{i=1}^{m}(x_i - y_i)$, and hence $x_1 = y_1, ..., x_m = y_m$, because of (b′).

Conversely, (b′) follows from (b) if we set $x = \mathbf{0}, x_1 = ... = x_m = \mathbf{0}$.It follows from condition (b) that every pair of subspaces $L_1, ..., L_m$ has only the element $\mathbf{0}$ in common. In fact, if $z \in L_j$ and $z \in L_k$, then using (b) and comparing the two expansions $z = z + \mathbf{0}, z \in L_j, \mathbf{0} \in L_k$ and $z = \mathbf{0} + z, \mathbf{0} \in L_j, \quad z \in L_k$, we find that $z = \mathbf{0}$.Thus an n-dimensional space V_n is the hyperfinite direct sum of the n one-dimensional subspaces determined by any n linearly independent vectors.

Moreover, the space V_n can be represented in various ways as a direct sum of subspaces not all of dimension 1.

Remark 2.2. Let L be a fixed subspace of an n-dimensional space V_n. Then there always exists a Subspace $M \subset V_n$ such that the whole space V_n is the direct sum of L and M.To prove this, we use the vectors $\mathbf{f}_{l+1}, \mathbf{f}_2, ..., \mathbf{f}_n$ constructed above, which are linearly independent over the subspace L. Let M be the subspace consisting of all linear combinations of the vectors $\mathbf{f}_{l+1}, \mathbf{f}_2, ..., \mathbf{f}_n$. Then M satisfies the stipulated requirement.

In fact, since the vectors $\mathbf{f}_1, \mathbf{f}_2, ..., \mathbf{f}_n$ form a basis in V_n, every vector $x \in L$ has an expansion of the form

$$\mathbf{x} = Ext\text{-}_{i=1}^{l} \alpha_i \mathbf{f}_i + Ext\text{-}_{i=l+1}^{n} \alpha_i \mathbf{f}_i = \mathbf{y} + \mathbf{z} \tag{2.24}$$

where

$$\mathbf{y} = Ext\text{-}_{i=1}^{l} \alpha_i \mathbf{f}_i \in L, \mathbf{z} = Ext\text{-}_{i=l+1}^{n} \alpha_i \mathbf{f}_i \in M. \tag{2.25}$$

Moreover $x = \mathbf{0}$ implies $\alpha_i = 0, 1 \leq i \leq n$, since the vectors $\mathbf{f}_i, 1 \leq i \leq n$ are linearly independent. Therefore conditions (a)-(b′) are satisfied, so that V_n is the direct sum of L and M.

Remark 2.3. If the dimension of the space L_k equals r_k, $1 \leq k \leq m$ and if r_k linearly independent vectors $\mathbf{f}_{k_1}, \mathbf{f}_2, ..., \mathbf{f}_{k_{r_k}}$ are selected in each space L_k, then every vector $\mathbf{x}$ of the sum $L = Ext\text{-}_{i=1}^{k} L_i$ can be expressed as a linear combination of these vectors.

Hence the dimension of the sum of the spaces $L_1, ..., L_k$ does not exceed the sum of the dimensions of the separate spaces. If the hyperfinite sum $Ext\text{-}_{i=1}^{k} L_i$ is direct, then the vectors $\mathbf{f}_{1_1}, ..., \mathbf{f}_{1_{r_1}} ..., \mathbf{f}_{k_1}, ..., \mathbf{f}_{m_1}, ... f_{m_{r_m}}$, are all linearly independent, so that in this case the dimension of the sum is precisely the hyperfinite sum of the dimensions.

Remark 2.4. In the general case, the dimension of the sum is related to the dimensions of the summands in a more complicated way. Here we consider only the problem of determining the dimension of the sum of two hyperfinite-dimensional subspaces P and Q of the space V, of dimensions p and q,respectively. Let L be the intersection of the subspaces P and Q, and let L have dimension l.

First we choose a basis $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_l$ in L. Then, using the argument mentioned above, we augment the basis $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_l$ by the vectors $\mathbf{f}_{l+1}, \mathbf{f}_{l+2}, ..., \mathbf{f}_p$ to make a basis for the whole subspace P and by the vectors $\mathbf{g}_{I+1}, \mathbf{g}_{I+2}, ..., \mathbf{g}_q$ to make a basis for the whole subspace Q. By definition, every vector in the sum $P + Q$ is the sum of a vector from P and a vector from Q, and hence can be expressed as a linear combination of the vectors

$$\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_l, \mathbf{f}_{l+1}, \mathbf{f}_{l+2}, ..., \mathbf{f}_p, \mathbf{g}_{I+1}, \mathbf{g}_{I+2}, ..., \mathbf{g} \tag{2.26}$$

We now show that these vectors form a basis for the subspace $P + Q$. To show this, it remains to verify their linear independence. Assume that there exists a linear relation of the form

$$Ext\text{-}_{i=1}^{l}\alpha_i\mathbf{e}_i + Ext\text{-}_{i=l+1}^{p}\beta_i\mathbf{f}_i + Ext\text{-}_{i=l+1}^{q}\gamma_i\mathbf{g}_i, \tag{2.27}$$

where at least one of the coefficients $\alpha_1, ..., \gamma_q$ is different from zero. We can then assert that at least one of the numbers $\gamma_{l+1}, ...\gamma_q$, is different from zero, since otherwise the vectors $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_l, \mathbf{f}_{l+1}, \mathbf{f}_{l+2}, ..., \mathbf{f}_p$ would be linearly dependent, which is impossible in view of the fact that they form a basis for the subspace P.

Consequently the vector

$$\mathbf{x} = Ext\text{-}_{i=l+1}^{q}\gamma_i\mathbf{g}_i \neq 0 \tag{2.28}$$

for otherwise the vectors $\mathbf{g}_{I+1}, \mathbf{g}_{I+2}, ..., \mathbf{g}_q$ would be linearly dependent. But it follows from (2.24) that

$$-x = Ext\text{-}_{i=1}^{l}\alpha_i\mathbf{e}_i + Ext\text{-}_{i=l+1}^{p}\beta_i\mathbf{f}_i \tag{2.29}$$

while (2.25) shows that $\mathbf{x} \in Q$. Thus $\mathbf{x}$ belongs to both P and Q, and hence belongs to the subspace L. But then

$$\mathbf{x} = Ext\text{-}_{i=l+1}^{q}\gamma_i\mathbf{g}_i = Ext\text{-}_{i=1}^{l}\lambda_i\mathbf{e}_i \tag{2.30}$$

and since the vectors $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_l, \mathbf{g}_{I+1}, \mathbf{g}_{I+2}, ..., \mathbf{g}_q$ are linearly independent, we have $\gamma_{l+1}, ..., \gamma_q = 0$. This contradiction shows that the vectors (2.23) are actually linearly independent, and hence form a basis for the subspace $P + Q$. It follows from Theorem 2.9 that the dimension of $P + Q$ equals the number of basis vectors (2.23). But this number equals $p + q - l$.

Theorem 2.11.[18].The dimension of the sum of two subspaces is equal to the sum of their dimensions minus the dimension of their intersection.

Corollary 2.1. Let V_p, and V_q, be two subspaces of dimensions p and q respectively, of an n-dimensional space $V_n, n \in \mathbb{N}^{\#}\backslash\mathbb{N}$, and suppose $p + q > n$. Then the intersection$V_p \cap V_q$ is of dimension no less than $p + q - n$.

2.4 Factor spaces

Definition 2.11.[18].(a) Given a subspace $\mathbf{L}$ of a linear space V, an element $y \in \mathbf{V}$ is said to be comparable with an element $443 \in \mathbf{V}$ (or comparable relative to $\mathbf{L}$) if $x - 443 \in \mathbf{L}$.

Obviously, if x is comparable with y, then 443 is comparable with x, so that the relation of comparability is symmetric. Every element $x \in \mathbf{V}$ is comparable with itself. Moreover, if x is comparable with 443 and 443 is comparable with z, then x is comparable with z, since $x - z = (x - y) + (y - z) \in \mathbf{L}$.

(b) The set of ail elements $443 \in \mathbf{V}$ comparable with a given element $x \in V$ is called a class, and is denoted by $[x]$. As just shown, a class $[x]$ contains the element y itself, and every pair of elements $443 \in [x], z \in [x]$ are comparable with each other. Moreover, if $u \neq [x]$, then 438 is not comparable with any element of $[x]$.

Therefore two classes either have no elements in common or else coincide completely. The subspace **L** itself is a class. This class is denoted by [0], since it contains the zero element of the space **V**.

(c) The whole space V can be partitioned into a set of nonintersecting classes $[x], [y], ...$

This set of classes will be denoted by $\mathbf{V}/\mathbf{L}$.

We now introduce linear operations in $\mathbf{V}/\mathbf{L}$ as follows: Given two classes $[x], [y]$ and two elements $\alpha, \beta$of the field ${}^*\mathbb{R}_c^{\#}$, we wish to define the class $\alpha[x]+\beta[y]$. To do this, we choose arbitrary elements $x_1 \in [x], y_1 \in [y]$ and find the class $[z]$ containing the element $z = \alpha x_1 + \beta y_1$. This class is then denoted by $\alpha[x] + \beta[y]$. Clearly, $\alpha[x]$ $+\beta[y]$ is uniquely defined. In fact, suppose we choose another element x_1 of the class $[x]$ and another ement y_1 of the class $[y]$. Then $(\alpha x_1 + \beta y_1) - (\alpha x + \beta y) = \alpha(x_1 - x) + \beta(y_1 - y)$ belongs to the space **L**, since $x_1 - x$ and $y_1 - y$ both belong to L. It follows that $\alpha x_1 + \beta y_1$ belongs to the same class as $\alpha x + \beta y$.

In particular, the above prescription defines addition of two classes $[x]$ and $[y]$, as well as multiplication of a class by a number $\alpha \in {}^*\mathbb{R}_c^{\#}$. We now show that these operations obey the axioms of a linear space, mentioned above. In fact, the validity of these axioms for classes follows at once from their validity for elements of the space V. Moreover, the zero element of the space $\mathbf{V}/\mathbf{L}$ is the class [0](consisting of all elements of the subspace **L**), while the inverse of the class $[x]$ is the class consisting of all inverses of elements of the class $[x]$. Thus all axioms are satisfied for the set of classes $\mathbf{V}/\mathbf{L}$. The resulting linear space $\mathbf{V}/\mathbf{L}$ is called the factor space of the space **V** with respect to the subspace **L**.

Theorem 2.12.[18]. Let $\mathbf{V} = \mathbf{V}_n, n \in \mathbb{N}^{\#}\backslash\mathbb{N}$ be an n-dimensional linear space over the field ${}^*\mathbb{R}_c^{\#}$,and let $\mathbf{L} = \mathbf{L}_l \subset \mathbf{V}$ be an l-dimensional subspace of **V**. Then the factor space $\mathbf{V}/\mathbf{L}$ is of dimension $n - l$.

Proof. Choose any basis $\mathbf{f}_1, \mathbf{f}_2, ..., \mathbf{f}_l \in \mathbf{L}$, and augment it, as mentioned above, by vectors $\mathbf{f}_{l+1}, \mathbf{f}_2, ..., \mathbf{f}_n$ to make a basis for the whole space V. Then the classes $[\mathbf{f}_{l+1}], [\mathbf{f}_2], ..., [\mathbf{f}_n]$ form a basis in the space $\mathbf{V}/\mathbf{L}$. To see this, we note that given any $x \in \mathbf{V}$, there is a representation

$$x = Ext\text{-}\sum_{k=1}^{n} \alpha_k \mathbf{f}_k, \tag{2.31}$$

and hence a representation

$$[x] = Ext\text{-}\sum_{k=l+1}^{n} \alpha_k [\mathbf{f}_k] \tag{2.32}$$

for the class $[x]$. Moreover, the classes $[\mathbf{f}_{l+1}], [\mathbf{f}_2], ..., [\mathbf{f}_n]$ are linearly independent.

In fact, if $Ext\text{-}\sum_{k=l+1}^{n} \alpha_k [\mathbf{f}_k] = [0] \in \mathbf{V}/\mathbf{L}$ for any $\alpha_k, 1 \leq n$ in ${}^*\mathbb{R}_c^{\#}$, then, in particular, there would be a relation $Ext\text{-}\sum_{k=l+1}^{n} \alpha_k [\mathbf{f}_k] \in L$.But $\mathbf{f}_{l+1}, \mathbf{f}_{l+2}$

2.5 Linear manifolds

An important way of constructing subspaces is to form the linear manifold spanned by a given hyperfinite system of vectors.

Definition 2.11.[18].Let $x_i, 1 \leq i \leq k, k \in \mathbb{N}^{\#}\backslash\mathbb{N}$ be a system of vectors of a linear space V.

Then the linear manifold spanned by $x_i, 1 \leq i \leq k$ is the set of all linear combinations

$$Ext\text{-}\sum_{i=1}^{k} \alpha_i x_i \tag{2.33}$$

with coefficients $\alpha_i, 1 \leq i \leq k$ in the field ${}^*\mathbb{R}_c^\#$.

It is easily verified that this set has properties (a) and (b) of Sect. 2.1.
Therefore the linear manifold spanned by a system $x_i, 1 \leq i \leq k$ is a subspace of the space V.

Obviously, every subspace containing the vectors $x_i, 1 \leq i \leq k$ also contains all their linear combinations (2.28). Consequently, the linear manifold spanned by the vectors $x_i, 1 \leq i \leq k$. is the smallest subspace containing these vectors.

The linear manifold spanned by the vectors $x_i, 1 \leq i \leq k$ is denoted by $\mathbf{L}\left(\{x_i\}_{i=1}^k\right)$.

Examples

(i) The linear manifold spanned by the basis vectors $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n, n \in \mathbb{N}^\# \backslash \mathbb{N}$ of a space V_n is obviously the whole space $\mathbf{V}_n$.

(ii) The linear manifold spanned by the system of functions $1, t, t^2 ..., t^n, n \in \mathbb{N}^\# \backslash \mathbb{N}$ consists of the set of all external hyper polynomials in the variable t with coefficients in the field ${}^*\mathbb{R}_c^\#$ of degree no higher than n.

(iii) The linear manifold spanned by the system of functions $1, t, t^2 ..., t^n, n \in \mathbb{N}^\# \backslash \mathbb{N}$ consists of the set of all external hyper polynomials in the variable t with coefficients in the field ${}^*\mathbb{C}_c^\#$ of degree no higher than n.

Lemma 2.1. If the vectors $\{x_i'\}_{i=1}^k$ belong to the linear manifold spanned by the vectors $\{x_i\}_{i=1}^k$, then the linear manifold $\mathbf{L}\left(\{x_i\}_{i=1}^k\right)$ contains the whole linear manifold $\mathbf{L}\left(\{x_i'\}_{i=1}^k\right)$.

Proof. Since the vectors $\{x_i'\}_{i=1}^k$ belong to the subspace $\mathbf{L}\left(\{x_i\}_{i=1}^k\right)$ then all their linear combinations, whose totality constitutes the linear manifold $\mathbf{L}\left(\{x_i'\}_{i=1}^k\right)$, also belong to the subspace of the $\mathbf{L}\left(\{x_i\}_{i=1}^k\right)$.

Lemma 2.2. Every vector of the system $\{x_i\}_{i=1}^k$ which is linearly dependent on the other vectors of the system can be eliminated without changing the linear manifold spanned by $\{x_i\}_{i=1}^k$.

Proof. If the vector x_1, say, is linearly dependent on the vectors $\{x_i\}_{i=2}^k$ this means that $x \in \mathbf{L}\left(\{x_i\}_{i=2}^k\right)$. It follows from Lemma 2.1 that $\mathbf{L}\left(\{x_i\}_{i=1}^k\right) \subset \mathbf{L}\left(\{x_i\}_{i=2}^k\right)$.

On the other hand, obviously $\mathbf{L}\left(\{x_i\}_{i=2}^k\right) \subset \mathbf{L}\left(\{x_i\}_{i=1}^k\right)$ .Together these two relations imply $\mathbf{L}\left(\{x_i\}_{i=1}^k\right) = \mathbf{L}\left(\{x_i\}_{i=2}^k\right)$.

We now will consider the problem of constructing a basis for a linear manifold and determining the dimension of a linear manifold. In solving this problem, we will assume that the number of vectors $\{x_i\}_{i=1}^k$ spanning the linear manifold $\mathbf{L}\left(\{x_i\}_{i=1}^k\right)$is hyperfinite, although some of our conclusions do not actually require this assumption.

Suppose that among the vectors $\{x_i\}_{i=1}^k$ spanning the linear manifold $\mathbf{L}\left(\{x_i\}_{i=1}^k\right)$ we can find $r \in \mathbb{N}^\#$ linearly independent vectors $\{\widetilde{x}_i\}_{i=1}^r$, say, such that every vector of the system $\{x_i\}_{i=1}^k$ is a linear combination of $\{\widetilde{x}_i\}_{i=1}^r$. Then the vectors $\{\widetilde{x}_i\}_{i=1}^r$ form a basis for the space $\mathbf{L}\left(\{x_i\}_{i=1}^k\right)$.

Indeed, by the very definition of a linear manifold, every vector z can be expressed as a linear combination of a hyperfinite number of vectors of the system $\{x_i\}_{i=1}^k$. But, by hypothesis, each of these vectors can be expressed as a linear combination of $\{\widetilde{x}_i\}_{i=1}^r$. Thus eventually the vector z can also be expressed as a linear combination of the vectors $\{\widetilde{x}_i\}_{i=1}^r$.

This, together with the assumption that the vectors $\{\widetilde{x}_i\}_{i=1}^r$ are linearly independent, shows that $\{\widetilde{x}_i\}_{i=1}^r$ indeed form a basis, as asserted.

According to Theorem 2.10, the dimension of the space $\mathbf{L}\left(\{x_i\}_{i=1}^k\right)$ is equal to the number r. Since there can be no more than r linearly independent vectors in an r-dimensional space, we get the following:

(a) If the number of vectors $\{x_i\}_{i=1}^k$ spanning $\mathbf{L}\left(\{x_i\}_{i=1}^k\right)$ is larger than the number r, then the vectors $\{x_i\}_{i=1}^k$ are linearly dependent.

If the number of these vectors equals r, then the vectors are linearly independent.

(b) Every set of $r+1$ vectors from the system $\{x_i\}_{i=1}^k$ is linearly dependent.

(c) The dimension of the space $\mathbf{L}\left(\{x_i\}_{i=1}^k\right)$ can be defined as the maximum number of linearly independent vectors in the system $\{x_i\}_{i=1}^k$.

3 Algebra of External Hyperfinite Polynomials

Definition 3.1.[18]. A linear space V over field ${}^*\mathbb{R}_c^\#$ or over field ${}^*\mathbb{C}_c^\# = {}^*\mathbb{R}_c^\# + i{}^*\mathbb{R}_c^\#$

is called an algebra over ${}^*\mathbb{R}_c^\#$ (or over ${}^*\mathbb{C}_c^\# = {}^*\mathbb{R}_c^\# + i{}^*\mathbb{R}_c^\#$) if there is defined on the elements $x, y, ...$ of V an operation of multiplication, denoted by xy, which satisfies the following conditions:

(1) $\alpha(xy) = (\alpha x)y = x(\alpha y)$ for every $x, 443 \in V$(or $\in {}^*\mathbb{C}_c^\#$) and every $\alpha \in {}^*\mathbb{R}_c^\#$ (${}^*\mathbb{C}_c^\#$);

(2) $(xy)z = x(yz)$ for every $x, y, z \in V$ (the associative law);

(3) $(x + y)z = xz + yz$ for every $x, y, z \in V$ (the distributive law).

In general, multiplication may not be commutative, i.e., we may have $xy \neq yx$.

Definition 3.2.[18].If multiplication is commutative, i.e., if

(4) $xy = yx$ for every $x, 443 \in V$, then the algebra V is said to be commutative.

Definition 3.3.[18].An element $e \in V$ is called a left unit if $ex = x$ for every $x \in V$, a right unit if $xe = x$ for every $x \in V$, and a two-sided unit or simply a unit if $ex = xe = x$ for every $x \in V$.

Definition 3.4.[18].An element $x \in V$ is called a left inverse of the element $443 \in V$ if xy is the unit of the algebra V; in this case, 443 is called a right inverse of x. If an element z has both a left and a right inverse, then the two inverses are unique and in fact coincide.

The element z is then said to be invertible, and its inverse is denoted by z^{-1}.
The product zu of an invertible element z and an invertible element u is an invertible element with inverse $u^{-1}z^{-1}$. If the element u is invertible, then the equation $u445 = v$ has the solution $x = u^{-1}v$. This solution is unique, being obtained by multiplying the equation $u445 = v$ on the left by u^{-1}.In the commutative case, we write $x = v/u$ or $x = v : u$, calling the element x the quotient of the elements v and u.

Definition 3.5.[18].An algebra V over field ${}^*\mathbb{R}_c^\#$ (${}^*\mathbb{C}_c^\#$) is said to have hyperfinite dimension n if V has dimension $n \in \mathbb{N}^\#\backslash\mathbb{N}$ regarded as a linear space. We will denote such algebra by V_n.

Example 3.1. An example of a nontrivial commutative algebra over a field ${}^*\mathbb{R}_c^\#$ (${}^*\mathbb{C}_c^\#$) is given by the set $\Pi^\#$ of all hyperfinite polynomials

$$P(\lambda) = Ext\text{-}_{k=0}^{m} a_k \lambda^k, \tag{3.1}$$

$m \in \mathbb{N}^{\#}\backslash\mathbb{N}$,with coefficients in ${}^{*}\mathbb{R}_{c}^{\#}$ (${}^{*}\mathbb{C}_{c}^{\#}$), equipped with the usual operations of addition and multiplication. This "polynomial algebra" has a unit, namely the polynomial $e(\lambda)$ with $a_0 = 1$ and all other coefficients equal to 0.

Example 3.2. The linear Space $M_n({}^{*}\mathbb{R}_{c}^{\#})$ of all matrices of order $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$ with elements in ${}^{*}\mathbb{R}_{c}^{\#}$, with the usual definition of matrix multiplication, is an example of a hyperfinite dimensional noncommutative algebra of dimension n^2.
Example 3.3.A more general example of a hyperfinite dimensional non-commutative algebra $L_n({}^{*}\mathbb{R}_{c}^{\#})$ with a unit is the linear space of all linear operators acting in a linear space $V_n, n \in \mathbb{N}^{\#}\backslash\mathbb{N}$ with the usual definition of operator multiplication.

Definition 3.6.[18].A subspace $L \subset V_n$ is called a subalgebra of the algebra V_n if $x \in L$, $y \in L$,implies $xy \in L$. A subspace $L \subset V_n$ is called a right ideal in V_n if $x \in L, 443 \in K$ implies $xy \in L$ and a left ideal in V_n if $x \in L, 443 \in K$ implies $yx \in L$. An ideal which is both a left and a right ideal is called a two-sided ideal. In a commutative algebra there is no distinction between left, right and two-sided ideals. There are two obvious two-sided ideals in every algebra V_n, i.e., the algebra V_n itself and the ideal $\{0\}$ consisting of the zero element alone. All other one-sided and two-sided ideals are called proper ideals.

Every ideal is a subalgebra, but the converse is in general false. Thus the set of all polynomials $P(\lambda)$ satisfying the condition $P(0) = P(1)$ is a subalgebra of the algebra Π which is not an ideal, while the set of all polynomials $P(\lambda)$ satisfying the condition $P(0) = 0$ is a proper ideal of the algebra Π.

Definition 3.7.[18].Let $L \subset V_n$ be a subspace of the algebra V_n, and consider the factor space V_n/L,i.e., the linear space consisting of the classes X of elements $x \in V_n$ which are comparable relative to L. If L is a two-sided ideal in V_n, then, besides linear operations, we can introduce an operation of multiplication for the classes $X \in V_n/L$. In fact, given two classes X and Y, choose arbitrary elements $x \in X, y \in Y$ and interpret XY as the class containing the product xy. This uniquely defines XY, since if $x' \in X, 443 \in Y$, then $xy' - xy = 445'(x-y) + (x-x)y$, and hence $x443 - xy$ belongs to L together with $y' - 443$ and $x' - x$. Moreover, since conditions (1)-(3) of Definition 3.1 hold in V_n, the analogous conditions hold for the classes $X \in V_n/L$. Therefore the factor space V_n/L equipped with the above operation of multiplication, is also an algebra, called the factor algebra of the algebra V_n with respect to the two-sided ideal L. If the algebra V_n is commutative, then obviously so is the factor algebra V_n/L.

Definition 3.8.[18].Let V_n' and V_n'' be two algebras over a field ${}^{*}\mathbb{R}_{c}^{\#}$ (${}^{*}\mathbb{C}_{c}^{\#}$).

Then a morphism ω of the space V_n' into the space V_n'' is called a morphism of the algebra V_n' into the algebra V_n'' if besides satisfying the two conditions:

(i) $\omega(x' + y') = \omega(x\prime) + \omega(y')$ for every $x', y' \in V_n'$,

(ii) $\omega(\alpha x') = \alpha\omega(x')$ for every $x' \in V_n'$ and every $\alpha \in {}^{*}\mathbb{R}_{c}^{\#}$ ($\alpha \in {}^{*}\mathbb{C}_{c}^{\#}$),

(iii) $\omega(x'y') = \omega(x')\omega(y')$ for every $x', y' \in V_n'$.

Remark 3.1. Let ω be a morphism of an algebra V_n' into an algebra V_n''. Then the set If of all vectors $x' \in V_n'$ such that $\omega(x') = 0$, which is obviously a subspace of V_n', is a two-sided ideal of the algebra V_n'.
In fact, if $x' \in L'$, $y' \in V_n'$, then $\omega(x'y') = \omega(x')\omega(y') = \mathbf{0}$, so that $x'y' \in L'$, and similarly $y'x' \in L'$, i.e., L' is a two-sided ideal of V_n', as asserted.

As in Remark 3.1 let Ω be the monomorphism of the space V_n'/L' into the space V_n'' which assigns to each class $X' \in V_n'/L'$ the (unique) element $\omega(x'), x' \in X'$. Then is a monomorphism of the algebra V_n'/L' into the algebra V_n''. In fact, choosing $x' \in X', y' \in Y'$,we have $x'y' \in X'Y'$ and

$$\Omega(\mathbf{X}'\mathbf{Y}') = \omega(x'y') = \omega(x')\omega(y') = \Omega(\mathbf{X}')\Omega(\mathbf{Y}'). \tag{3.2}$$

If the morphism ω is an epimorphism of the algebra V' into the algebra V'', then the morphism Ω is an isomorphism of the algebra V'/L' onto the algebra V''.

Let $\mathbf{A}$ be a linear operator acting in a space V over a field ${}^*\mathbb{C}_c^{\#}$. Since addition and multiplication by constants in ${}^*\mathbb{C}_c^{\#}$ are defined for linear operators acting in V, with every polynomial $P(\lambda) = Ext\text{-}_{k=0}^{m} a_k \lambda^k$ we can associate the operator

$$P(\mathbf{A}) = Ext\text{-}_{k=0}^{m} a_k \mathbf{A}^k \tag{3.3}$$

acting in the same space V as $\mathbf{A}$ itself. Then the rule associating $P(\lambda)$ with $P(\mathbf{A})$ has the properties: (1) if

$$P(\lambda) = P_1(\lambda) + P_2(\lambda) = Ext\text{-}_{k=0}^{m} a_k \lambda^k + Ext\text{-}_{k=0}^{m} b_k \lambda^k = Ext\text{-}_{k=0}^{m} (a_k + b_k)\lambda^k, \tag{3.4}$$

then

$$P(\mathbf{A}) = Ext\text{-}_{k=0}^{m} (a_k + b_k)\mathbf{A}^k = Ext\text{-}_{k=0}^{m} a_k \mathbf{A}^k + Ext\text{-}_{k=0}^{m} b_k \mathbf{A}^k = P_1(\mathbf{A}) + P_2(\mathbf{A}). \tag{3.5}$$

Similarly (2) if

$$Q(\lambda) = P_1(\lambda) P_2(\lambda) = \left(Ext\text{-}_{i=0}^{m} a_i \lambda^i\right)\left(Ext\text{-}_{k=0}^{m} b_k \lambda^k\right) = Ext\text{-}_{i=0}^{m}\left(Ext\text{-}_{k=0}^{m} a_i b_k \lambda^{i+k}\right), \tag{3.6}$$

then

$$Q(\mathbf{A}) = Ext\text{-}_{i=0}^{m}\left(Ext\text{-}\ _{k=0}^{m} a_i b_k \mathbf{A}^{i+k}\right) = P_1(\mathbf{A}) P_2(\mathbf{A}) \tag{3.7}$$

by the distributive law for operators.
Note that the operators $P_1(A)$ and $P_2(A)$ always commute with each other, regardless of the choice of the polynomials $P_1(X)$ and $P_2(X)$.

The resulting morphism of the algebra $\Pi^{\#}$ of polynomials into the algebra $L_n({}^*\mathbb{R}_c^{\#})$ of linear operators acting in V_n (Example 3.3) is in general not an epimorphism, if only because operators of the form $P(\mathbf{A})$ commute with each other, while the whole algebra $L_n({}^*\mathbb{R}_c^{\#})$ is noncommutative.

There exists an isomorphism between the algebra $L_n({}^*\mathbb{R}_c^{\#})$ of all linear operators acting in the n-dimensional space V_n and the algebra $M_n({}^*\mathbb{R}_c^{\#})$ of all matrices of order n with elements from the field ${}^*\mathbb{R}_c^{\#}$.

This isomorphism is established by fixing a basis $e_1 ..., e_n$ in the space V_n and assigning for every operator $\mathbf{A} \in M_n({}^*\mathbb{R}_c^{\#})$ its matrix in this basis. Both algebras $L_n({}^*\mathbb{R}_c^{\#})$ and $M_n({}^*\mathbb{R}_c^{\#})$ have the same hyperfinite dimension n^2.

The set of all hyperfinite polynomials of the form $P(\lambda)Q_0(\lambda)$, where $Q_0(\lambda)$ is a fixed polynomial and $P(\lambda)$ an arbitrary polynomial, is obviously an ideal in the commutative algebra $\Pi^{\#}$ of all polynomials $P(\lambda)$ with coefficients in a field ${}^*\mathbb{R}_c^{\#}$ $({}^*\mathbb{C}_c^{\#})$

(Example 3.1).

Conversely, we now show that every ideal $\mathbf{I} \neq \{0\}$ of the algebra $\Pi^{\#}$ is of this structure, i.e., is obtained from some polynomial $Q_0(\lambda)$ by multiplication by an arbitrary polynomial hyperfinite $P(\lambda)$. To this end, we find the nonzero polynomial of lowest degree, say q, in the ideal $\mathbf{I}$, and denote it by $Q_0(\lambda)$. We then assert that every polynomial $Q(\lambda)$ in $\mathbf{I}$ is of the form $P(\lambda)Q_0(\lambda)$, where $P(\lambda) \in \Pi^{\#}$. In fact, as is familiar from elementary algebra,

$$Q(\lambda) = P(\lambda)Q_0(\lambda) + R(\lambda) \tag{3.8}$$

where $R(\lambda)$ is the quotient obtained by dividing $Q(\lambda)$ by $Q_0(\lambda)$ and $P(\lambda)$ is the remainder, of degree less than the divisor $Q_0(\lambda)$, i.e., less than the number q. But the polynomials $Q(\lambda)$ and $Q_0(\lambda)$ belong to the ideal $\mathbf{I}$, and hence, as is apparent from (3.7), so does the remainder $P(\lambda)$. Since the degree of $P(\lambda)$ is less than q and since $Q_0(\lambda)$ has the lowest degree, namely q, of all nonzero polynomials in $\mathbf{I}$, it follows that $P(\lambda) = 0$, and the requireed assertion is proved.The polynomial $Q_0(\lambda)$ is said to generate the ideal $\mathbf{I}$.

Remark 3.2. The polynomial $Q_0(\lambda)$ is uniquely determined by the ideal $\mathbf{I}$ to within a numerical factor. In fact, if the polynomial $Q_1(\lambda)$ has the same property as the polynomial $Q_0(\lambda)$, then, as just shown, $Q_1(\lambda) = P_1(\lambda)Q_0(\lambda), Q_n(\lambda) = P_0(\lambda)Q_1(\lambda)$.

It follows that the degrees of the polynomials $Q_1(\lambda)$ and $Q_0(\lambda)$ coincide and that $P_1(\lambda)$ and $P_0(\lambda)$ do not contain λ and hence are numbers, as asserted.

Remark 3.3. Given polynomials $Q_1(\lambda), ..., Q_m(\lambda)$ not all equal to zero and with no common divisors of degree $> I$, we now show that there exist polynomials $P_1^0(\lambda), ..., P_m^0(\lambda)$ such that

$$Ext\text{-}_{i=0}^{m} P_i^0(\lambda)Q_i(\lambda) \equiv 1. \tag{3.9}$$

In fact, let $\mathbf{I}$ be the set of all polynomials of the form

$$Ext\text{-}_{i=0}^{m} P_i^0(\lambda)Q_i(\lambda) \tag{3.10}$$

with arbitrary $P_1(\lambda), ..., P_m(\lambda)$ in $\Pi^{\#}$. Then $\mathbf{I}$ is obviously an ideal in $\Pi^{\#}$.
In particular

$$Q_1(\lambda) = S_1(\lambda)G_0(\lambda), ..., Q_m(\lambda) = S_m(\lambda)G_0(\lambda), \tag{3.11}$$

where $S_1(\lambda), ..., S_m(\lambda)$ are certain polynomials, from which it follows that $Q_0(\lambda)$ is a common divisor of the polynomials $Q_1(\lambda)$, ..., $Q_m(\lambda)$.

But, by hypothesis, the degree of $Q_0(\lambda)$ is zero, and hence $Q_0(\lambda)$ is a constant a_0, where $a_0 \neq 0$ since otherwise $\mathbf{I} = \{0\}$.

Multiplying (3.9) by 430_0^{-1} and writing $P_k^0(\lambda) = \widetilde{P}_k^0(\lambda)430_0^{-1}$, we get (3.8), as required.

4 Canonical form of the Matrix of an Arbitrary Operator

Let $\mathbf{A}$ denote an arbitrary linear operator acting in an n-dimensional space $V_n, n \in \mathbb{N}^{\#} \backslash \mathbb{N}$
Since the operations of addition and multiplication are defined for such operators , with every hyperfinite external polynomial

$$P\left(\lambda\right)=Ext\text{-}_{k=0}^{m}a_{k}\lambda^{k} \tag{4.1}$$

we can associate an operator

$\mathrm{P}(\mathbf{A})=Ext\text{-}_{k=0}^{m}a_{k}\mathbf{A}^{k}(4.2)$
acting in the same space V_n, where addition and multiplication of polynomials corresponds to addition and multiplication of the associated operators in the sense of Sect. 3. In fact, if

$\mathrm{P}(\lambda)=P_{1}\left(\lambda\right)+P_{2}\left(\lambda\right)=Ext\text{-}_{k=0}^{m}a_{k}\lambda^{k}+Ext\text{-}_{k=0}^{m}b_{k}\lambda^{k}=Ext\text{-}_{k=0}^{m}\left(a_{k}+b_{k}\right)\lambda^{k},(4.3)$
then
$\mathrm{P}(\mathbf{A})=Ext\text{-}_{k=0}^{m}\left(a_{k}+b_{k}\right)\mathbf{A}^{k}=Ext\text{-}_{k=0}^{m}a_{k}\mathbf{A}^{k}+Ext\text{-}_{k=0}^{m}b_{k}\mathbf{A}^{k}=P_{1}\left(\mathbf{A}\right)+P_{2}\left(\mathbf{A}\right).(4.4)$
Similarly, if

$$\begin{gathered}Q\left(\lambda\right)=P_{1}\left(\lambda\right)P_{2}\left(\lambda\right)=\left(Ext\text{-}_{i=0}^{m}a_{i}\lambda^{i}\right)\left(Ext\text{-}_{k=0}^{m}b_{k}\lambda^{k}\right)=\\ Ext\text{-}_{i=0}^{m}\left(Ext\text{-}_{k=0}^{m}a_{i}b_{k}\lambda^{i+k}\right),\end{gathered} \tag{4.5}$$

then

$$Q\left(\mathbf{A}\right)=Ext\text{-}_{i=0}^{m}\left(Ext\text{-}_{k=0}^{m}a_{i}b_{k}\mathbf{A}^{i+k}\right)=P_{1}\left(\mathbf{A}\right)P_{2}\left(\mathbf{A}\right) \tag{4.6}$$

by the distributive law for operator multiplication. In particular, the operators $P_1(\mathbf{A})$ and $P_2(\mathbf{A})$ always commute.

Thus the mapping $\omega(P(\lambda))=P(\mathbf{A})$ is an epimorphism of the algebra $\Pi^{\#}$ of all hyperfinite polynomials with coefficients in the field ${}^{*}\mathbb{R}_{c}^{\#}$ (${}^{*}\mathbb{C}_{c}^{\#}$) into the algebra$\Pi_{\mathbf{A}}^{\#}$ of all linear operators of the form $P(\mathbf{A})$ acting in the space V_n. By Sect.3, the algebra $\Pi_{\mathbf{A}}^{\#}$ is isomorphic to the factor algebra $\Pi_{\mathbf{A}}^{\#}/\mathbf{I}_{A}$, where $\mathbf{I}_A$ is the ideal consisting of all polynomials $P(\lambda)$ such that $\omega(P(\lambda))=P(\mathbf{A})=0$.

We now analyze the structure of this ideal. As noted in Example 3.3, the set of all linear operators acting in a space $V_n, n\in\mathbb{N}^{\#}\backslash\mathbb{N}$ is an algebra of hyperfinite dimension n^2 over the field ${}^{*}\mathbb{R}_{c}^{\#}$ (${}^{*}\mathbb{C}_{c}^{\#}$). Hence, given any operator $\mathbf{A}$, it follows that the first n^2+1 terms of the hyperfinite sequence $\mathbf{A}_0=\mathbf{E},\mathbf{A},\mathbf{A}^2,,,.,\mathbf{A}^m,$...must be linearly dependent. Suppose that

$$Ext\text{-}_{k=0}^{m}a_{k}\mathbf{A}^{k}=0, \tag{4.7}$$

where $m\leq n^2$. Then, by the correspondence between polynomials and operators mentioned above the hyperfinite polynomial

$$Q\left(\lambda\right)=Ext\text{-}_{k=0}^{m}a_{k}\lambda^{k} \tag{4.8}$$

must correspond to the zero operator. Every polynomial $Q\left(\lambda\right)$ for which the operator $Q(\mathbf{A})$ is the zero operator is called an annihilating polynomial of the operator $\mathbf{A}$, Thus we have just shown that every operator $\mathbf{A}$ has an annihilating polynomial of degree $\leq n^2$.The set of all annihilating polynomials of the operator $\mathbf{A}$ is an ideal in the algebra $\Pi^{\#}$. By sect. 3 there is a polynomial $Q_0\left(\lambda\right)$ uniquely determined to within a numerical factor such that all annihilating polynomials are of the form $P\left(\lambda\right)Q_0(\lambda)$ where $P\left(\lambda\right)$ is an arbitrary polynomial in $\Pi^{\#}$, In particular, $Q_0(\lambda)$ is the annihilating polynomial of lowest degree among all annihilating polynomials of the operator $\mathbf{A}$.

Hence $Q_0(\lambda)$ is called the minimal annihilating polynomial of the operator $\mathbf{A}$.

Theorem 4.1.[18]. Let $Q(\lambda)$ be an annihilating polynomial of the operator $\mathbf{A}$, and suppose that $Q(\lambda) = Q_1(\lambda)Q_2(\lambda)$,where the factors $Q_1(\lambda)$ and $Q_2(\lambda)$ are relatively prime. Then the space V_n can be represented as the direct sum $V_n = \mathbf{T}_1 \oplus \mathbf{T}_2$ of two subspaces $\mathbf{T}_1$ and $\mathbf{T}_2$ both invariant with respect to the operator $\mathbf{A}$, where $Q_1(\mathbf{A})x_2 = 0, Q_2(\mathbf{A})x_1 = 0$ for arbitrary $x_1 \in \mathbf{T}_1, x_2 \in \mathbf{T}_2$, so that $Q_1(\lambda)$ and $Q_2(\lambda)$ are annihilating polynomials for the operator $\mathbf{A}$ acting in the subspaces $\mathbf{T}_2$ and $\mathbf{T}_1$, respectively.

Proof. By sect.3 there exist polynomials $P_1(\lambda)$ and $P_2(\lambda)$ such that

$$P_1(\lambda)Q_1(\lambda) + P_2(\lambda)Q_2(\lambda) \equiv 1, \tag{4.9}$$

and hence

$$P_1(\mathbf{A})Q_1(\mathbf{A}) + P_2(\mathbf{A})Q_2(\mathbf{A}) \equiv \mathbf{E}. \tag{4.10}$$

Let $\mathbf{T}_k$,$k = 1, 2$ denote the range of the operator $Q_k(\mathbf{A})$, i.e., the set of all vectors of the form $Q_k(\mathbf{A})x, x \in V_n$.Then obviously $y = Q_k(\mathbf{A})x \in \mathbf{T}_k$ implies $\mathbf{A}y = Q_k(\mathbf{A})\mathbf{A}x \in \mathbf{T}_k$, so that the subspace $\mathbf{T}_k$ is invariant with respect to the operator $\mathbf{A}$, Given any $x_1 \in \mathbf{T}_1$, there is a vector $y \in V_n$ such that $Q_2(\mathbf{A})x_1 = Q_1(\mathbf{A})Q_2(\mathbf{A})z = Q(\mathbf{A})z = \mathbf{0}$, and similarly, given any $x_2 \in \mathbf{T}_2$,there is a vector $z \in V_n$ such that $Q_1(\mathbf{A})x_2 = Q_1(\mathbf{A})Q_2(\mathbf{A})z = Q(\mathbf{A})z = \mathbf{0}$.

Moreover, given any $x \in V_n$, we have $x = Q_1(\mathbf{A})P_1(\mathbf{A})x + Q_2(\mathbf{A})P_2(\mathbf{A})x = x_1 + x_2$,where $x_k = Q_k(\mathbf{A})P_k(\mathbf{A})x \in \mathbf{T}_k$, $k = 1, 2$.

It follows that V_n is the sum of the subspaces $\mathbf{T}_1$ and $\mathbf{T}_2$. If $x_0 \in \mathbf{T}_1 \cap \mathbf{T}_2$, then $Q_1(\mathbf{A})x_0 = Q_2(\mathbf{A})x_0 = \mathbf{0}$, and hence $x_0 = P_1(\mathbf{A})Q_1(\mathbf{A})x_0 + P_2(\mathbf{A})Q_2(\mathbf{A})x_0 = 0$.

Therefore $\mathbf{T}_1 \cap \mathbf{T}_2 = \{0\}$, and the sum $V_n = \mathbf{T}_1 \oplus \mathbf{T}_2$ is direct.

Remark 4.1. By construction, the operator $Q_1(\mathbf{A})$ annihilates the subspace $\mathbf{T}_2$, while the operator $Q_2(\mathbf{A})$ annihilates the subspace $\mathbf{T}_1$, We now show that every vector x annihilated by the operator $Q_1(\mathbf{A})$ belongs to $\mathbf{T}_2$, while every vector x annihilated by the operator $Q_2(\mathbf{A})$ belongs to $\mathbf{T}_1$.

In fact, suppose $Q_1(\mathbf{A})x = 0$. We have $x = x_1 + x_2$,where $x_1 \in \mathbf{T}_1, x_2 \in \mathbf{T}_2$, and hence $Q_1(\mathbf{A})x_1 = Q_1(\mathbf{A})x - Q_1(\mathbf{A})x_2 = 0$ since $Q_1(\mathbf{A})x_2 = 0$. But $Q_2(\mathbf{A})x_1 = 0$ as well, since $x_1 \in \mathbf{T}_1$. It follows that $x_1 = P_1(\mathbf{A})Q_1(\mathbf{A})x_1 + P_2(\mathbf{A})Q_2(\mathbf{A})x_1 = \mathbf{0}, x = x_2 \in \mathbf{T}_2$.

Similarly, $Q_2(\mathbf{A})x = 0$ implies $x \in \mathbf{T}_1$, and our assertion is proved.

Remark 4.2. Representing the polynomials $Q_1(\lambda)$ and $Q_2(\lambda)$ themselves as products of further prime factors, we can decompose the space V_n into smaller subspaces invariant with respect to the operator $\mathbf{A}$ and annihilated by the appropriate factors of $Q_1(\lambda)$ and $Q_2(\lambda)$. Suppose the annihilating polynomial $Q(\lambda)$ has a factorization of the form

$$Q(\lambda) = Ext\text{-}_{k=1}^{m} (\lambda - \lambda_k)^{r_k}\,, \tag{4.11}$$

$where\lambda_1, ..., \lambda_m$ are all the (distinct) roots of $Q(\lambda)$ and r_k is the multiplicity of λ_k. For example, such a factorization is always possible (to within a numerical factor) in the field ${}^*\mathbb{C}_c^\#$.

Theorem 4.2.[18]. Suppose the operator $\mathbf{A}$ has an annihilating polynomial of the form (4.11). Then the space $V_n, n \in \mathbb{N}^\# \backslash \mathbb{N}$ can be represented as the direct sum

$$V_n =_{k=1}^{m} T_k \tag{4.12}$$

of m subspaces $T_1, ..., T_m$, all invariant with respect to $\mathbf{A}$, where the subspace T_k is annihilated by $\mathrm{B}_k^{r_k}$, the r_k-th power of the operator $\mathbf{B}_k = \mathbf{A} - \lambda_k \mathbf{E}$.

Proof. Apply Theorem **4.1** repeatedly to the factorization (4.11) of $Q(\lambda)$ into m relatively prime factors of the form $(\lambda - \lambda_j)^{r_j}$.

Theorem 4.3.[18]. Let V_ω be countable dimentional subspace of the space $V_n, n \in \mathbb{N}^\# \backslash \mathbb{N}$ and $\mathbf{A}\left(V_n \backslash V_\omega\right) = \mathbf{0}$.

Suppose that the operator $\mathbf{A}$ has an annihilating polynomial of the form

$$Q_\omega(\lambda) = Ext\text{-}_{k=1}^{m} Q_\omega(\lambda, \lambda_k) = Ext\text{-}_{k=1}^{m}\left[Ext\text{-}\left(\lambda - \lambda_k\right)^\omega\right], \tag{4.13}$$

$m \in \mathbb{N}^\#$,where the function $Q_\omega(\lambda, \lambda_k)$ is defined by the following formula

$$Q_\omega(\lambda, \lambda_k) = Ext\text{-}_{i=1}^{p} \Theta_i\left(\lambda, \lambda_k\right), \tag{4.14}$$

where $p \in \mathbb{N}^\# \backslash \mathbb{N}$, $\Theta_i\left(\lambda, \lambda_k\right) = \left(\lambda - \lambda_k\right)$ for all $i \in \mathbb{N}$ and $\Theta_i\left(\lambda, \lambda_0\right) \equiv 1$ for all $i \in \mathbb{N}^\# \backslash \mathbb{N}$.

Then the space V_ω, can be represented as the direct sum

$$V_\omega =_{k=1}^{m} T_k \tag{4.15}$$

of $m \in \mathbb{N}^\#$ subspaces $T_1, ..., T_m$, all invariant with respect to $\mathbf{A}$, where the subspace T_k is annihilated by $\mathbf{B}_k^\omega$ the ω-th power of the operator $\mathbf{B}_k = \mathbf{A} - \lambda_k \mathbf{E}$. Here the operator $\mathbf{B}_k^\omega$ is defined by

$$\mathbf{B}_k^\omega = Ext\text{-}_{i=1}^{p} \mathbf{B}_{k,i}, \tag{4.16}$$

where $p \in \mathbb{N}^\# \backslash \mathbb{N}$, $\mathbf{B}_{k,i} = \mathbf{B}_k$ for all $i \in \mathbb{N}$ and $\mathbf{B}_{k,i} \equiv \mathbf{1}$ for all $i \in \mathbb{N}^\# \backslash \mathbb{N}$.

Proof. Apply Theorem 4.17.1 repeatedly to the factorization (4.17.10) of $Q(\lambda)$ into m relatively prime factors of the form $Q_\omega(\lambda, \lambda_k)$.

5 The Solution of the Invariant Subspace Problem

The proof for convenience of the readers divided in 5 parts (as in author paper [18]), see Subsections 5.1-5.5.

5.1 Stage I. Embedding $l_2 \hookrightarrow l^{\#}_{2,\omega} \hookrightarrow V_n, n \in \mathbb{N}^{\#}/\mathbb{N}$

An classical sequence space that is a subspace of the set of all sequences real or complex numbers $x = (x_1, x_2, ...) = (x_i)_{i=1}^{\infty}$.

The set of all $\mathbb{R}$-valued (or $\mathbb{C}$-valued) countable sequences we denote by l_ω.

For any $k \in \mathbb{N}$ let $\mathbf{e}_k = \mathbf{e}_k[i]$ be the sequence defined by

$$\mathbf{e}_k[i] = \begin{cases} 1 & \text{if} \quad i = k \\ 0 & \text{if} \quad i \neq k \end{cases} (5.1)$$

The space l_2 (of square-summable sequences of real or complex numbers) is the set of infinite sequences of real or complex numbers such that

$$\|x\|_2 = \sum_{i=1}^{\infty} |x_i|^2 < \infty. \quad (5.2)$$

l_2 is isomorphic to all separable, infinite dimensional Hilbert spaces.

There is a canonical countable basis $\{\mathbf{e}_k\}_{k=1}^{\infty}$ such that $\mathbf{e}_1 = (1, 0, ...)$, $\mathbf{e}_2 = (0, 1, ...)$,etc.

Remark 5.1.1. Note that there is the canonical embeding $r \to {}^*r$ [10]:

$$\mathbb{R} \hookrightarrow_* {}^*\mathbb{R}(5.3)$$

and we denote emadge of this embeding by

$${}^*\mathbb{R}_{\mathbf{st}} \subset {}^*\mathbb{R}(5.4)$$

Thus we replace (5.1.2) by

$$\|{}^*x\|_2 = \sum_{i=1}^{\infty} |{}^*x_i|^2 < \infty. \quad (5.5)$$

Definition 5.1.1. The space $l_{2,\omega}$ is the set of all ${}^*\mathbb{R}_{\mathbf{st}}$-valued (or ${}^*\mathbb{C}_{\mathbf{st}}$-valued) countable sequences such that:

$$\|{}^*x\|_{2,\omega} = Ext\text{-}\sum_{i=1}^{\omega} |{}^*x_i|^2 < \infty. \quad (5.6)$$

Remark 5.1.2. Note that in general case $\|{}^*x\|_2 \neq \|{}^*x\|_{2,\omega}$,since in general case:

$$\sum_{i=1}^{\infty} |{}^*x_i|^2 \neq Ext\text{-}\sum_{i=1}^{\omega} |{}^*x_i|^2 < \infty, \quad (5.7)$$

see subsect. 3.12.

Remark 5.1.3. The set of all ${}^*\mathbb{R}_c^{\#}$-valued (or ${}^*\mathbb{C}EndExpansion_c^{\#}$-valued) countable sequences we denote by $l_{2,\omega}^{\#}$.

Definition 5.1.2. Let $V_n, n \in \mathbb{N}^{\#}/\mathbb{N}$ be a hyperfinite-dimensional vector space over external non-Archimedian field ${}^*\mathbb{R}_c^{\#}$.Such vector space consists of all external and internal ${}^*\mathbb{R}_c^{\#}$-valued hyperfinite sequences (called a vector) $\mathbf{x} = \{x_i\}_{i=1}^{i=n} = \{x_i\}_{i\in n}$ of hyperreal numbers, called the coordinates or components of vector $\mathbf{x}$.The vector sum of $\mathbf{x} = \{x_i\}_{i=1}^{i=n}$ and $\mathbf{y} = \{y_i\}_{i=1}^{i=n}$ is

$$\mathbf{x} + \mathbf{y} = \{x_i + y_i\}_{i=1}^{i=n}. \tag{5.8}$$

If $a \in \mathbb{R}_c^{\#}$ is a hyperreal number, the scalar multiple of $\mathbf{x}$ by $a$is

$$a \times \mathbf{x} = \{a \times x_i\}_{i=1}^{i=n}. \tag{5.9}$$

there is a canonical hyperfinite basis $\{\mathbf{e}_i'\}$ $1 \le i \le n, n \in \mathbb{N}^{\#}/\mathbb{N}$ such that $\mathbf{e}_1' = ({}^*1, {}^*0, ...), \mathbf{e}_2 = ({}^*0, {}^*1, ...), ...,$ see subsect. 3.1.

Remark 5.1.4. Note that there is the natural embedding $l_\omega^{\#} \hookrightarrow V_n, n \in \mathbb{N}^{\#}/$

5.2 Stage II.Extending of the bounded operator $A : l_2 \to l_2$ up to operator $\widehat{A} : V_n \to V_n$.

Remind that bounded operators $A : l_2 \to l_2$ admit matrix representations completely analogous to the well known matrix representations of operators on finite dimensional spaces [20].

We choose any orthonormal basis $\{\mathbf{e}_k\}_{k=1}^{\infty}$ in l_2 and let $Ae_k = c_k \in l_2$,

$$(Ae_k, e_i) = a_{ik} \tag{5.10}$$

and therefore

$$c_k = \sum_{i=1}^{\infty} a_{ik} e_i, \tag{5.11}$$

where $\sum_{i=1}^{\infty} |a_{ik}|^2 < \infty, k \in \mathbb{N}$. We introduce the infinite matrix $\mathbf{A}_\omega$

$$\begin{matrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ . & . & . & \cdots \end{matrix} \tag{5.12}$$

of which the elements of the k-th column are the components of the vector into which the operator A maps the Ath coordinate vector. If the operator A is bounded, then it is uniquely determined by the infinite matrix (a_{ik}) or the proof of this assertion it is necessary to show how to represent (he operator in terms of the matrix and the orthonormal basis $\{e_k\}_{k=1}^{\infty}$. Thus, we have:

$$Ae_k = \sum_{i=1}^{\infty} a_{ik} e_i. \tag{5.13}$$

Since the operator A is linear, it is well defined on the linear envelope of (he given basis, i.e., for all vectors each of which has only a finite number of nonzero components relative to the basis. Since $\mathbf{A}$ is continuous, the value of $\mathbf{A}f$ for an arbitrary vector $f \in l_2$ may be found by the limit.It is not difficult to write a simple formula for the components of the vector $\mathbf{A}f$ indeed, if

$$f = \sum_{k=1}^{\infty} x_k \mathbf{e}_k, \tag{5.14}$$

then

$$\mathbf{A}f = \sum_{k=1}^{\infty} y_k \mathbf{e}_k, \tag{5.15}$$

where

$$y_k = \sum_{i=1}^{\infty} a_{ki} x_i \tag{5.16}$$

Definition 5.2.1. If the operator $A : l_2 \to l_2$ is defined everywhere in l_2 and if its value for any vector (5.2.5) is given by the formulas (5.2.6) and (5.2.7), then we say that the operator A admits a matrix representation relative to the orthogonal basis $\{e_k\}_{k=1}^{\infty}$.

Theorem 5.2.1. Every bounded linear operator $A : l_{22}$ defined on the entire space admits a matrix representation with respect to each orthogonal basis.

Proof. Let $f_n = \sum_{k=1}^{n} x_k \mathbf{e}_k$,then $Af_n = \sum_{k=1}^{\infty} y_k^{(n)} \mathbf{e}_k$,where $y_k^{(n)} = \sum_{k=1}^{n} a_{ki} x_i$.By the boundedness of the operator A, we get

$$y_k = (Af, \mathbf{e}_k) = \lim_{n\to\infty} (Af_n, \mathbf{e}_k) = \lim_{n\to\infty} y_k^{(n)} = \lim_{n\to\infty} \sum_{k=1}^{n} a_{ki} x_i = \sum_{k=1}^{\infty} a_{ki} x_i. \tag{5.17}$$

Theorem 5.2.2. If an operator $A : l_2 \to l_2$ defined everywhere in a separable space l_2, admits a matrix representation (5.2.3) with respect to some orthogonal basis $\{\mathbf{e}_k\}_{k=1}^{\infty}$, then it is bounded.

Proof. By hypothesis, the series $(Af, e_k) = \sum_{i=1}^{\infty} a_{ki} x_i$ converges for each vector $= \sum_{k=1}^{\infty} x_k \mathbf{e}_k$,where $\{\mathbf{e}_k\}_{k=1}^{\infty}$ the orthonormal basis, mentioned in the theorem, with respect to which the operator A admits a matrix representation. Therefore, by the theorem of Landau (see [20] Section 18), one obtains :

$$\sum_{i=1}^{\infty} |a_{ki}|^2 < \infty, k \in \mathbb{N} \tag{5.18}$$

We introduce the sequence of vectors $c_k^* = \sum_{i=1}^{\infty} \overline{a}_{ki} e_i, k \in \mathbb{N}$ and by means of them, define the linear operator A^*. First, let $A^* \mathbf{e}_k = c_k^*$ and then use linearity to define A^* on the linear envelope of the set of vectors $\mathbf{e}_k$. Finally, extend A^* by continuity to all of l_2. It is easy to prove that for any $f, g \in l_2, (Af, g) = (f, A^* g)$ after which, to complete the proof, it remains to apply Hellinger and Toeplitz theorem, see [20] Section 26.
Remark 5.2.1. In view of the inequality (5.2.9), the expression

$$\Phi_k(f) = \sum_{i=1}^{\infty} a_{ki} x_i, k \in \mathbb{N} \tag{5.19}$$

defines a linear functional of $f = \sum_{k=1}^{\infty} x_k \mathbf{e}_k$ and therefore, $P_n(f) = \sqrt{\sum_{k=1}^{n} \Phi_k^2(f)}, n \in \mathbb{N}$ defines a convex continuous functional of f. Since the sequence $\{P_n(f)\}_{n=1}^{\infty}$ is bounded for each $f \in l_2$. On the basis of the corollary of the lemma concerning convex functionals [20], the functional

$$P(f) = \sup_{n \in \mathbb{N}} P_n(f) = \lim_{n \to \infty} P_n(f) = \sqrt{\sum_{k=1}^{\infty} \Phi_k^2(f)} = \|Af\| \tag{5.20}$$

is continuous, i.e., there exists a constant M such that $P(f) \leq M\|f\|$,but this implies that the operator A is bounded.

Remark 5.2.2. The proof of the theorem can be formulated also in the following form: if for arbitrary numbers $x_k, k \in \mathbb{N}$ such that:

$$\sum_{k=1}^{\infty} |x_k|^2 < \infty \tag{5.21}$$

the inequality

$$\sum_{k=1}^{\infty} \left| \sum_{i=1}^{\infty} a_{ki} x_i \right|^2 < \infty \tag{5.22}$$

holds, then there exists a constant M such that:

$$\sum_{k=1}^{\infty} \left| \sum_{i=1}^{\infty} a_{ki} x_i \right|^2 \leq M^2 \sum_{k=1}^{\infty} |x_k|^2 . \tag{5.23}$$

Definition 5.2.2. A sequence $\{x_k\}_{k=1}^{\infty} \in l_2$ is admissible sequence if

$$\sum_{k=1}^{\infty} |x_k|^2 = Ext\text{-}_{k=1}^{\omega} |x_k|^2 . \tag{5.24}$$

Definition 5.2.3. Let $\{\mathbf{e}_k\}_{k=1}^{\infty}$ be Schauder basis in l_2,i.e.,the standard unit vector basis in l_2.Vector $f = \sum_{k=1}^{\infty} x_k \mathbf{e}_k \in l_2$ is admissible vector of l_2 if sequence $\{x_k\}_{k=1}^{\infty} \in l_2$ is admissible sequence.

Remark 5.2.3. Note that if vector $f = \sum_{k=1}^{\infty} x_k \mathbf{e}_k$ is admissible vector of l_2 then $f = \sum_{k=1}^{\infty} x_k \mathbf{e}_k = Ext\text{-}\sum_{k=1}^{\omega} x_k \mathbf{e}_k$.

Definition 5.2.4. We extend now the operator $A : l_2 \to l_2$ is given on l_2 by the Eq.(5.2.6)-Eq.(5.2.7) up to the operator $\widehat{A} :^{\#} \to^{\#}$ is given on $^{\#}$ by the ω-sum is given by the Eq.(5.2.16)-Eq.(5.2.17)

$$\widehat{A} f = Ext\text{-}\sum_{k=1}^{\omega} \widehat{y}_k \mathbf{e}_k, \tag{5.25}$$

where

$$\widehat{y}_k = Ext\text{-}\sum_{i=1}^{\omega} a_{ki} x_i. \tag{5.26}$$

Theorem 5.2.3. Assume that a sequence $\{x_k\}_{k=1}^{\infty} \in l_2$ is admissible sequence and let $f = \sum_{k=1}^{\infty} x_k \mathbf{e}_k \in l_2$. Let $\{\widehat{y}_k\}_{k=1}^{\infty}$ be a sequence is given by the Eq.(5.2.17).

Then (i) sequence $\{\widehat{y}_k\}_{k=1}^{\infty}$ is admissible sequence and (ii) $\{\widehat{y}_k\}_{k=1}^{\infty} \in l_2$
Proof. If sequence $\{x_k\}_{k=1}^{\infty} \in l_2$ is admissible sequence, then by Definition 5.2.2 we get

$$\sum_{k=1}^{\infty} |x_k|^2 = Ext\text{-}_{k=1}^{\omega} |x_k|^2 . \tag{5.27}$$

From the equality (5.2.18) by Theorem 3.12.6 in [18] we obtain

$$\lim_{m\to\infty} Ext\text{-}_{k=m}^{\omega} |x_k|^2 = 0. \tag{5.28}$$

From the equality (5.2.17) we get for all $k \in \mathbb{N}$:

$|\widehat{y}_k| \leq Ext\text{-} \sum_{i=1}^{\omega} |a_{ki}x_i| \leq \alpha_k \left(Ext\text{-} \sum_{i=1}^{\omega} |x_i|^2\right), (5.29)$
where $\alpha_k = Ext\text{-}\sum_{i=1}^{\omega} |a_{ki}|^2 < \infty$.We set now $x_i = 0, i \leq m$ in (5.2.20) and therefore

$$|\widehat{y}_{k,m}| = \left| Ext\text{-} \sum_{i=m}^{\omega} a_{ki}x_i \right| \leq Ext\text{-} \sum_{i=m}^{\omega} |a_{ki}x_i| \leq \alpha_k \left(Ext\text{-} \sum_{i=m}^{\omega} |x_i|^2 \right), \tag{5.30}$$

From the inequality (5.2.21) by the equality (5.2.19) for the all $k \in \mathbb{N}$ we get

$$\lim_{m\to\infty} |\widehat{y}_{k,m}| = \lim_{m\to\infty} \left| Ext\text{-} \sum_{i=m}^{\omega} a_{ki}x_i \right| \leq \lim_{m\to\infty} \left(Ext\text{-} \sum_{i=m}^{\infty} |x_k|^2 \right) = 0 \tag{5.31}$$

and therefore

$$\lim_{m\to\infty} |\widehat{y}_{k,m}| = 0. \tag{5.32}$$

From the equality (5.2.18) by Theorem 3.12.6 in [18] we get for all $k \in \mathbb{N}$ that $\widehat{y}_k = Ext\text{-} \sum_{i=m}^{\omega} a_{ki}x_i = \sum_{i=m}^{\infty} a_{ki}x_i = y_k$.(5.33)
Thus we get for all $k \in \mathbb{N}$ that $\widehat{y}_k = y_k$.
Theorem 5.2.4. Assume that a vector $f = \sum_{k=1}^{\infty} x_k \mathbf{e}_k$ is admissible vector of l_2 then for all $n \in \mathbb{N}$ vector $\left(\widehat{\mathbf{A}}\right)^n f$ is admissible vector of l_2.

Proof. Immediately by Definition 5.2.3 from Theorem 5.2.3.

5.3 The operator $\widehat{A} : V_n \to V_n$.

Using the canonical imbedding $\mathbb{R} \hookrightarrow_* {}^*\mathbb{R}$ (5.1.3) we imbed now the infinite matrix $\mathbf{A}_\omega = (a_{ij})_{i,j\in\mathbb{N}}$(5.2.3) into the infinite matrix $\mathbf{A}^*_\omega = ({}^*a_{ij})_{i,j\in\mathbb{N}}$ (5.2.25) such that

$$\mathbf{A}_{\omega}^{*}=\left\Vert\begin{array}{ccccc} {}^{*}a_{11} & {}^{*}a_{12} & \cdots & {}^{*}a_{1n} & \cdots \\ {}^{*}a_{21} & {}^{*}a_{22} & \cdots & {}^{*}a_{2n} & \cdots \\ . & . & . & . & \cdots \\ {}^{*}a_{n1} & {}^{*}a_{n2} & \cdots & {}^{*}a_{nn} & \cdots \\ . & . & \cdots & . & . \\ . & . & \cdots & . & \cdots \\ . & . & \cdots & . & \cdots \end{array}\right\Vert \tag{5.34}$$

where ${}^{*}a_{ij}\in{}^{*}\mathbb{R}_{\mathbf{st}}$, $i,j\in\mathbb{N}$.

We imbed now the infinite matrix $\mathbf{A}_{\omega}^{*}=({}^{*}a_{ij})_{i,j\in\mathbb{N}}$ (5.2.3) into the hyperfinite matrix $\mathbf{A}_{\omega,\mathbf{m}}^{*}=({}^{*}a_{ij})_{i\leq\mathbf{m},j\leq m}$,where $\mathbf{m}\in\mathbb{N}^{\#}\setminus\mathbb{N}$ and for $i,j\in\mathbb{N}^{\#}\setminus\mathbb{N}$ the conditions (i)-(ii) are satisfied (i) ${}^{*}a_{ij}={}^{*}0$ if $i\neq j$, (ii) ${}^{*}a_{ij}={}^{*}1$ if $i=j$.

Thus $\mathbf{A}_{\omega,\mathbf{m}}^{*}$ is hyperfinite external matrix of the following literal form

$$\mathbf{A}_{\omega,\mathbf{m}}^{*}=\left\Vert\begin{array}{cccccccc} {}^{*}a_{11} & {}^{*}a_{12} & \cdots & {}^{*}a_{1n} & \cdots & {}^{*}0 & {}^{*}0 & \cdots \\ {}^{*}a_{21} & {}^{*}a_{22} & \cdots & {}^{*}a_{2n} & \cdots & {}^{*}0 & {}^{*}0 & \cdots \\ . & . & . & . & \cdots & {}^{*}0 & {}^{*}0 & \cdots \\ {}^{*}a_{n1} & {}^{*}a_{n2} & \cdots & {}^{*}a_{nn} & \cdots & {}^{*}0 & {}^{*}0 & \cdots \\ . & . & \cdots & . & \cdots & {}^{*}0 & {}^{*}0 & \cdots \\ {}^{*}0 & {}^{*}0 & \cdots & {}^{*}0 & \cdots & {}^{*}1 & {}^{*}0 & \cdots \\ . & . & \cdots & . & \cdots & {}^{*}0 & {}^{*}1 & \cdots \\ {}^{*}0 & {}^{*}0 & \cdots & {}^{*}0 & {}^{*}0 & {}^{*}0 & {}^{*}0 & {}^{*}1 \end{array}\right\Vert \tag{5.35}$$

where ${}^{*}a_{ij}\in{}^{*}\mathbb{R}_{\mathbf{st}}$, $1\leq i\leq\mathbf{m},1\leq j\leq\mathbf{m}$.The matrix $\mathbf{A}_{\omega,\mathbf{m}}^{*}$ is defined an external linear operator $\widehat{A}$ on $V_{\mathbf{m}}$ by the formula

$$\widehat{A}f=$$
$$\left\Vert\begin{array}{cccccccc} {}^{*}a_{11} & {}^{*}a_{12} & \cdots & {}^{*}a_{1n} & \cdots & {}^{*}0 & {}^{*}0 & \cdots \\ {}^{*}a_{21} & {}^{*}a_{22} & \cdots & {}^{*}a_{2n} & \cdots & {}^{*}0 & {}^{*}0 & \cdots \\ . & . & . & . & \cdots & {}^{*}0 & {}^{*}0 & \cdots \\ {}^{*}a_{n1} & {}^{*}a_{n2} & \cdots & {}^{*}a_{nn} & \cdots & {}^{*}0 & {}^{*}0 & \cdots \\ . & . & \cdots & . & \cdots & {}^{*}0 & {}^{*}0 & \cdots \\ {}^{*}0 & {}^{*}0 & \cdots & {}^{*}0 & \cdots & {}^{*}1 & {}^{*}0 & \cdots \\ . & . & \cdots & . & \cdots & {}^{*}0 & {}^{*}1 & \cdots \\ {}^{*}0 & {}^{*}0 & \cdots & {}^{*}0 & {}^{*}0 & {}^{*}0 & {}^{*}0 & {}^{*}1 \end{array}\right\Vert \times \left\Vert\begin{array}{c} x_1 \\ x_2 \\ \\ x_n \\ . \\ . \\ x_{m-1} \\ x_m \end{array}\right\Vert = \left\Vert\begin{array}{c} y_1 \\ y_2 \\ \\ y_n \\ . \\ . \\ y_{m-1} \\ y_m \end{array}\right\Vert \tag{5.36}$$

where $f=Ext\text{-}\sum_{k=1}^{m}x_k\mathbf{e}_k\in V_n$ and where the multiplication $\times$ is defined by

$$y_i=Ext\text{-}_{k=1}^{m}\,{}^{*}a_{ik}\,x_k, \tag{5.37}$$

where $1\leq i\leq\mathbf{m}$, see subsection 4.4.

5.4 Stage III.Proof that operator $\widehat{\mathbf{A}}$ has a non-trivial infinite-dimensional invariant subspaces $\Delta_1,\Delta_2\subsetneqq l_{2,\omega}^{\#}\subset V_{\mathbf{m}}$

Theorem 5.3.1. Suppose the operator $\mathbf{A}$ has an annihilating polynomial of the form

$$Q(\lambda)=Ext\text{-}_{k=1}^{m}\left[Ext\text{-}(\lambda-\lambda_k)^{r_k}\right], \tag{5.38}$$

where: (i) the function $Ext\text{-}(\lambda - \lambda_k)^{r_k}$,for all $r_k \in \mathbb{N}^{\#}$ is defined by the following formula

$$Ext\text{-}\left(\lambda - \lambda_k\right)^{r_k} = Ext\text{-}_{i=1}^{p}\Theta_i\left(\lambda, \lambda_k\right), \tag{5.39}$$

where $r_k < p \in \mathbb{N}^{\#}\backslash\mathbb{N}$, $\Theta_i\left(\lambda, \lambda_k\right) = \left(\lambda - \lambda_k\right)$ for all $1 \leq i \leq r_k$ and $\Theta_i\left(\lambda, \lambda_k\right) \equiv 1$ for all $i > r_k$, and (ii) the function $Ext\text{-}(\lambda - \lambda_k)^{r_k}$,for $r_k = \omega$ is defined by the formula

$$Ext\text{-}\left(\lambda - \lambda_k\right)^{r_k} = Ext\text{-}_{i\in\mathbb{N}^{\#}}\Theta_i\left(\lambda, \lambda_k\right), \tag{5.40}$$

with $\Theta_i\left(\lambda, \lambda_k\right) = \left(\lambda - \lambda_k\right)$ for all $i \in \mathbb{N}$ and $\Theta_i\left(\lambda, \lambda_k\right) \equiv 1$ for all $i \in \mathbb{N}^{\#}\backslash\mathbb{N}$.In this case we denote it by

$$Ext\text{-}\left(\lambda - \lambda_k\right)^{\omega} = Ext\text{-}_{i=1}^{\omega}\Theta_i\left(\lambda, \lambda_k\right). \tag{5.41}$$

Here $\lambda_1, ..., \lambda_m$ are all the (distinct) roots of $Q(\lambda)$ and r_k is the multiplicity of λ_k. For example, such a factorization is always possible (to within a numerical factor) in the field ${}^*\mathbb{C}_c^{\#}$ [18],see Appendix A.

Remark 5.3.1. Then the external linear space $V_{\mathbf{m}}, \mathbf{m} \in \mathbb{N}^{\#}\backslash\mathbb{N}$ can be represented as the direct sum

$$V_{\mathbf{m}} = Ext\text{-}_{k=1}^{r}T_k \tag{5.42}$$

of r subspaces $T_1, ..., T_r, r \in \mathbb{N}^{\#}$ all invariant with respect to $\mathbf{A}$, where the subspace T_k is annihilated by $\mathbf{B}_k^{r_k}$, the r_k -th power of the operator $\mathbf{B}_k = \mathbf{A}-\ \lambda_k\mathbf{E}$,see subsection 4.17.2 in [18] and [17].

The all vectors $\mathbf{x} \in V_{\mathbf{m}}$ has the form $\mathbf{x} = \{x_i\}_{i=1}^{i=\mathbf{m}}$.Thus we can represent the vector $\mathbf{x} \in V_{\mathbf{m}}$ as the sum $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$,where

$$\mathbf{x}_1 = \left\{x_i^{(1)}\right\}_{i\in\mathbb{N}} \tag{5.43}$$

and

$$\mathbf{x}_2 = \left\{x_i^{(2)}\right\}_{i\in\mathbb{N}^{\#}\backslash\mathbb{N}} \tag{5.44}$$

with $i \leq \mathbf{m}$.Then the space $V_{\mathbf{m}}$ can be represented as the direct sum

$$V_{\mathbf{m}} = V_{\mathbf{m}}^{(1)}V_{\mathbf{m}}^{(2)}, \tag{5.45}$$

where the subspace $V_{\mathbf{m}}^{(1)}$ contains the all vectors of the form $\mathbf{x}_1 = \left\{x_i^{(1)}\right\}_{i\in\mathbb{N}}$ and the subspace $V_{\mathbf{m}}^{(2)}$ contains the all vectors of the form $\mathbf{x}_2 = \left\{x_i^{(2)}\right\}_{i\in\mathbb{N}^{\#}\backslash\mathbb{N}}$ with $i \leq \mathbf{m}$.

The hyperfinite matrix $\mathbf{A}^*_{\omega,\mathbf{m}}$ is a direct sum $\mathbf{A}^*_{\omega,\mathbf{m}} = \mathbf{A}^*_{\omega}\mathbf{D}$,where $\mathbf{D}$ is infinite diagonal matrix $\mathbf{D} = diag$ $\left({}^*0, {}^*0, ..., {}^*1, {}^*1, ..., {}^*1\right)$.

Thus operator $\widehat{\mathbf{A}}$ has the following form: $\widehat{\mathbf{A}} = \widehat{\mathbf{A}}_1 + \widehat{\mathbf{A}}_2$,where $\widehat{\mathbf{A}}_1 = \widehat{\mathbf{A}} \upharpoonright V_{\mathbf{m}}^{(1)}$ and $\widehat{\mathbf{A}}_2 = \widehat{\mathbf{A}} \upharpoonright V_{\mathbf{m}}^{(2)}$.

We will be consider now the following three possible cases.

I.There is no invariant subspace of $V_{\mathbf{m}}^{(1)}$ with respect to $\widehat{\mathbf{A}}_1$.In this case we obtain $Q(\lambda) = Ext\text{-}(\lambda - \lambda_1)^{\omega}$,where $\lambda_1 \in \mathbb{R}_c^{\#}$.It follows from Theorem 1.5.4 in [18] that $\lambda_1 \in {}^*\mathbb{R}_{\mathbf{st}}$ and infinite matrice $\mathbf{A}_{\omega}^{*}(5.2.25)$ is diagonal.Thus in this case operator $\mathbf{A} : l_2 \to l_2$ has a form :

$$\mathbf{A} = \lambda_1 \times \mathbf{1}. \tag{5.46}$$

II.There is countable set of subspaces $\{T_k\}_{k\in\mathbb{N}}$ all invariant with respect to $\widehat{\mathbf{A}}_1$, where for all $k \in \mathbb{N}, \dim T_k < \infty$ and where

$$V_m^{(1)} = Ext\text{-}_{k\in\mathbb{N}} T_k. \tag{5.47}$$

Let $\mathbb{N}_1$and $\mathbb{N}_2$ be a subsets of $\mathbb{N}$ such that $\mathbb{N}_1\cup\ \mathbb{N}_2 = \mathbb{N}$ and $\mathbb{N}_1\cap\ \mathbb{N}_2 = \varnothing$.Now we choose an countable set of subspaces $\{T_i\}_{i\in\mathbb{N}_1} \subsetneqq \{T_k\}_{k\in\mathbb{N}}$ such that

$$\{T_k\}_{k\in\mathbb{N}} = \{T_i\}_{i\in\mathbb{N}_1} \cup \{T_j\}_{j\in\mathbb{N}_2}, \tag{5.48}$$

where $\{T_j\}_{j\in\mathbb{N}_2} = \{T_k\}_{k\in\mathbb{N}} \setminus \{T_i\}_{i\in\mathbb{N}_1}$ .Let $\Delta_1$and Δ_2 be a subspaces of $V_m^{(1)}$ such that

$$\Delta_1 = Ext\text{-}_{k\in\mathbb{N}_1} T_k \tag{5.49}$$

and

$$\Delta_2 = Ext\text{-}_{k\in\mathbb{N}_2} T_k \tag{5.50}$$

correspondingly and therefore

$$V_m^{(1)} = \Delta_1 \oplus \Delta_2. \tag{5.51}$$

III.There is only finite set of subspaces $\{T_k\}_{k\leq r}, r \in \mathbb{N}$ all invariant with respect to $\widehat{\mathbf{A}}_1$, where for all $1 \leq k \leq r, \dim T_k = \infty$ and where

$$V_m^{(1)} =_{k=1}^{r} T_k. \tag{5.52}$$

IV.There is countable set of subspaces $\{T_k\}_{k\in\mathbb{N}}$ all invariant with respect to $\widehat{\mathbf{A}}_1$, where for all $k \in \mathbb{N}, \dim T_k = \infty$ and where

$$V_m^{(1)} =_{k=1}^{\infty} T_k. \tag{5.53}$$

5.5 Stage IV.Proof that there exists an admissible vector $\Psi \in l_2 \wedge \Psi \in l_{2,\omega}^{\#}$

Theorem 5.3.4. There exists an admissible vector Ψ_1 in subspace Δ_1 and there exists an admissible vector Ψ_2 in subspace Δ_2.

Proof. Note that subspace Δ_1 has an countable basis $\left\{\mathbf{b}_i^{(1)}\right\}_{i=1}^{\infty}$,
where

$$\begin{array}{c} \mathbf{b}_1^{(1)} = Ext\text{-}\sum_{k_1=1}^{\omega} b_{k_1 1}^{(1)} \mathbf{e}_{k_1}, \\ \mathbf{b}_2^{(1)} = Ext\text{-}\sum_{k_1=1}^{\omega} b_{k_1 2}^{(1)} \mathbf{e}_{k_1}, \\ \cdots \\ \mathbf{b}_l^{(1)} = Ext\text{-}\sum_{k_1=1}^{\omega} b_{k_1 l}^{(1)} \mathbf{e}_{k_1}, \\ \cdots \end{array} \tag{5.54}$$

and similarly subspace Δ_2 has an countable basis $\left\{\mathbf{b}_{k_1}^{(2)}\right\}_{k_1=1}^{\infty}$, where

$$\begin{array}{c} \mathbf{b}_1^{(2)} = Ext\text{-}\sum_{k_1=1}^{\omega} b_{k_1 1}^{(1)} \mathbf{e}_{k_1}, \\ \mathbf{b}_2^{(2)} = Ext\text{-}\sum_{k_1=1}^{\omega} b_{k_1 2}^{(1)} \mathbf{e}_{k_1}, \\ \cdots \\ \mathbf{b}_l^{(2)} = Ext\text{-}\sum_{k_1=1}^{\omega} b_{k_1 l}^{(1)} \mathbf{e}_{k_1}, \\ \cdots \end{array} \tag{5.55}$$

and $\left\{\mathbf{b}_{k_1}^{(1)}\right\}_{k_1=1}^{\infty} \left\{\mathbf{b}_{k_1}^{(2)}\right\}_{k_1=1}^{\infty} = \varnothing$,where $\{k_1\}_{k_1=1}^{\infty} \cap \{k_2\}_{k_2=1}^{\infty} = \varnothing$ and $\{k_1\}_{k_1=1}^{\infty} \cup \{k_2\}_{k_2=1}^{\infty} = \mathbb{N}$.

We represent now basis vectors $\mathbf{b}_i^{(1)}, i = 1, 2, ...$ as infinite columns $\left(\mathbf{b}_i^{(1)}\right)^{\top}, i = 1, 2, ...$ of the following literal form

$$\left(\mathbf{b}_1^{(1)}\right)^{\top} = \left\| \begin{array}{c} b_{11}^{(1)} \\ b_{21}^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ b_{k_1 1}^{(1)} \\ b_{k_1+11}^{(1)} \\ \cdot \\ \cdot \\ \cdot \end{array} \right\|, \left(\mathbf{b}_2^{(1)}\right)^{\top} = \left\| \begin{array}{c} b_{12}^{(1)} \\ b_{22}^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ b_{k_1 2}^{(1)} \\ b_{k_1+12}^{(1)} \\ \cdot \\ \cdot \\ \cdot \end{array} \right\|, ..., \left(\mathbf{b}_{k_1}^{(1)}\right)^{\top} = \left\| \begin{array}{c} b_{1k_1}^{(1)} \\ b_{2k_1}^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ b_{k_1 k_1}^{(1)} \\ b_{k_1+1k_1}^{(1)} \\ \cdot \\ \cdot \\ \cdot \end{array} \right\| \tag{5.56}$$

Remark 5.4.1. Note that the infinite columns $\left(\mathbf{b}_i^{(1)}\right)^{\top}, i = 1, 2, ...$are linearly independent.

Similarly we represented basis vectors $\mathbf{b}_i^{(2)}, i = 1, 2, ...$ as infinite columns $\left(\mathbf{b}_i^{(1)}\right)^{\top}, i = 1, 2, ...$of the following literal form

$$\left(\mathbf{b}_1^{(2)}\right)^{\top} = \left\| \begin{array}{c} b_{11}^{(2)} \\ b_{21}^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ b_{k1}^{(2)} \\ b_{k+11}^{(2)} \\ \cdot \\ \cdot \\ \cdot \end{array} \right\|, \left(\mathbf{b}_2^{(2)}\right)^{\top} = \left\| \begin{array}{c} b_{12}^{(2)} \\ b_{22}^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ b_{k_2 2}^{(2)} \\ b_{k_2+12}^{(2)} \\ \cdot \\ \cdot \\ \cdot \end{array} \right\|, ..., \left(\mathbf{b}_{k_2}^{(2)}\right)^{\top} = \left\| \begin{array}{c} b_{1k}^{(2)} \\ b_{2k}^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ b_{k_2 k_2}^{(2)} \\ b_{k_2+1k_2}^{(2)} \\ \cdot \\ \cdot \\ \cdot \end{array} \right\| \tag{5.57}$$

Remark 2.4.1. Note that $\left\{\left(\mathbf{b}_{k_1}^{(1)}\right)^{\top}\right\}_{k_1=1}^{\infty} \left\{\left(\mathbf{b}_{k_2}^{(2)}\right)^{\top}\right\}_{k_2=1}^{\infty} = \varnothing$,since

$\left\{\mathbf{b}_{k_1}^{(1)}\right\}_{k_1=1}^{\infty}\left\{\mathbf{b}_{k_2}^{(2)}\right\}_{k_2=1}^{\infty}=\varnothing.$

Using the columns (5.4.3) we formed the following infinite matrix of the following literal form

$$\mathbf{\Omega}_{1,\omega}=\left\|\begin{array}{cccccccccc} b_{11}^{(1)} & b_{12}^{(1)} & \cdot & \cdot & \cdot & b_{1k}^{(1)} & b_{1k_1+1}^{(1)} & \cdot & \cdot & \cdot \\ b_{21}^{(1)} & b_{22}^{(1)} & \cdot & \cdot & \cdot & b_{2k}^{(1)} & b_{2k_1}^{(1)} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{k_11}^{(1)} & b_{k_12}^{(1)} & \cdot & \cdot & \cdot & b_{k_1k_1}^{(1)} & b_{k_1k_1+1}^{(1)} & \cdot & \cdot & \cdot \\ b_{k_1+11}^{(1)} & b_{k_1+12}^{(1)} & \cdot & \cdot & \cdot & b_{k_1+1k_1}^{(1)} & b_{k_1+1k_1+1}^{(1)} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right\| \tag{5.58}$$

And similarly using columns (5.4.4) we formed the following infinite matrix of the following literal form

$$\mathbf{\Omega}_{2,\omega}=\left\|\begin{array}{cccccccccc} b_{11}^{(2)} & b_{12}^{(2)} & \cdot & \cdot & \cdot & b_{1k}^{(2)} & b_{1k_2+1}^{(2)} & \cdot & \cdot & \cdot \\ b_{21}^{(2)} & b_{22}^{(2)} & \cdot & \cdot & \cdot & b_{2k}^{(2)} & b_{2k_2}^{(2)} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{k_21}^{(2)} & b_{k_22}^{(2)} & \cdot & \cdot & \cdot & b_{k_2k_2}^{(2)} & b_{k_2k_2+1}^{(2)} & \cdot & \cdot & \cdot \\ b_{k_2+11}^{(2)} & b_{k_2+12}^{(2)} & \cdot & \cdot & \cdot & b_{k_2+1k}^{(2)} & b_{k_2+1k_2+1}^{(2)} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \tag{5.59}$$

It follows directly from Theorem 4.14.2, see subsection 4.14, that

$$\det \mathbf{\Omega}_{1,\omega}\neq 0 \tag{5.60}$$

and

$$\det \mathbf{\Omega}_{2,\omega}\neq 0. \tag{5.61}$$

We conseder now the following infinite system of the linear equations \

$$\left\|\begin{array}{cccccccccc} b_{11}^{(1)} & b_{12}^{(1)} & \cdot & \cdot & \cdot & b_{1k}^{(1)} & b_{1k_1+1}^{(1)} & \cdot & \cdot & \cdot \\ b_{21}^{(1)} & b_{22}^{(1)} & \cdot & \cdot & \cdot & b_{2k}^{(1)} & b_{2k_1}^{(1)} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{k_11}^{(1)} & b_{k_12}^{(1)} & \cdot & \cdot & \cdot & b_{k_1k_1}^{(1)} & b_{k_1k_1+1}^{(1)} & \cdot & \cdot & \cdot \\ b_{k_1+11}^{(1)} & b_{k_1+12}^{(1)} & \cdot & \cdot & \cdot & b_{k_1+1k_1}^{(1)} & b_{k_1+1k_1+1}^{(1)} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}\right\|\times\left\|\begin{array}{c} y_1^{(1)} \\ y_2^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ y_{k_1}^{(1)} \\ y_{k_1+1}^{(1)} \\ \cdot \\ \cdot \\ \cdot \end{array}\right\|=\left\|\begin{array}{c} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{k_1} \\ x_{k_1+1} \\ \cdot \\ \cdot \\ \cdot \end{array}\right\| \tag{5.62}$$

and the following infinite system of the linear equations

$$\left\Vert \begin{array}{cccccccc} b_{11}^{(2)} & b_{12}^{(2)} & \cdot & \cdot & \cdot & b_{1k}^{(2)} & b_{1k_2+1}^{(2)} & \cdot\ \cdot\ \cdot \\ b_{21}^{(2)} & b_{22}^{(2)} & \cdot & \cdot & \cdot & b_{2k}^{(2)} & b_{2k_2}^{(2)} & \cdot\ \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot \\ b_{k_21}^{(2)} & b_{k_22}^{(2)} & \cdot & \cdot & \cdot & b_{k_2k_2}^{(2)} & b_{k_2k_2+1}^{(2)} & \cdot\ \cdot\ \cdot \\ b_{k_2+11}^{(2)} & b_{k_2+12}^{(2)} & \cdot & \cdot & \cdot & b_{k_2+1k}^{(2)} & b_{k_2+1k_2+1}^{(2)} & \cdot\ \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot \end{array} \right\Vert \times \left\Vert \begin{array}{c} y_1^{(2)} \\ y_2^{(2)} \\ \cdot \\ \cdot \\ \cdot \\ y_k^{(2)} \\ y_{k_2+1}^{(2)} \\ \cdot \\ \cdot \\ \cdot \end{array} \right\Vert = \left\Vert \begin{array}{c} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{k_2} \\ x_{k_2+1} \\ \cdot \\ \cdot \\ \cdot \end{array} \right\Vert \tag{5.63}$$

where a sequences $\{x_{k_1}\}_{k_1=1}^{\infty} \in l_2$ and $\{x_{k_2}\}_{k_2=1}^{\infty} \in l_2$are admissible sequences such that $\{x_{k_1}\}_{k_1=1}^{\infty} \cap \{x_{k_2}\}_{k_2=1}^{\infty} = \varnothing$, see Definition 2.2.2.

It follows directly from (5.4.5) by Theorem 4.12.2 in [18],see also [17], that the system (5.4.7) has a unique solution namely $\left\{\overline{y}_{k_1}^{(1)}\right\}_{k_1=1}^{\infty}$ and similarly it follows directly from (5.4.7) by Theorem 5.12.2 that the system (5.4.9) has the unique namely $\left\{\overline{y}_{k_2}^{(2)}\right\}_{k_2=1}^{\infty}$ .Thus finally we conclude that there is an admissible vector Ψ_1 :

$$\Psi_1 = Ext\text{-}\sum_{k=1}^{\omega} x_k \mathbf{e}_k \tag{5.64}$$

such that

$$\Psi_1 = \sum_{k_1=1}^{\infty} x_{k_1}\mathbf{e}_{k_1} = Ext\text{-}\sum_{k_1=1}^{\omega} x_{k_1}\mathbf{e}_{k_1} \in \Delta_1 \tag{5.65}$$

and there is an admissible vector Ψ_2 :

$$\Psi_2 = Ext\text{-}\sum_{k_2=1}^{\omega} x_{k_2}\mathbf{e}_{k_2} \tag{5.66}$$

such that

$$\Psi_2 = \sum_{k_2=1}^{\infty} x_{k_2}\mathbf{e}_{k_2} = Ext\text{-}\sum_{k_2=1}^{\omega} x_{k_2}\mathbf{e}_{k_2} \in \Delta_2. \tag{5.67}$$

Note that the statements (5.4.12) and (5.4.14) finalized the proof of Theorem 5.3.4.

5.6 Stage V. proof the main result by a contradiction

Assumption 5.5.1. We assume now that the a bounded linear operator $\mathbf{A} : l_2 \to l_2$ hasn't non trivial closed invariant subspace.

Let Ψ_1 be any admissible vector $\Psi_1 \in l_2$ given by Eq.(5.4.12) and let Ψ_2 be any admissible vector $\Psi_2 \in l_2$ given by Eq.(5.4.14).

It follows by Theorem 5.5.4 that for all $n \in \mathbb{N}$ for all $n \in \mathbb{N}$ vector $\left(\widehat{\mathbf{A}}\right)^n \Psi_1$ is admissible vector of l_2.

Proposition 5.5.1. Let Θ be any vector $\Theta \in l_2$ and let $\{c_i\}_{i=1}^{\infty}$ be $\mathbb{R}$-valued sequence such that

$$\Theta = \sum_{i=1}^{\infty} c_i f_i. \tag{5.68}$$

Where

$$f_i = \left(\widehat{\mathbf{A}}\right)^i \Psi_1, i \in \mathbb{N} \tag{5.69}$$

and where a series in RHS of the Eq.(5.5.1) converges absolutely in the norm $\|\cdot\|_2$ and therefore

$$\sum_{i=1}^{\infty} |c_i| \, \|f_i\|_2 < \infty. \tag{5.70}$$

Let $s_n, n \in \mathbb{N}$ be a partial sum

$$s_n = \sum_{i=1}^{n} c_i f_i. \tag{5.71}$$

There is a subsequence $\{c_{\nu_m}\}_{m=1}^{\infty} \subsetneqq \{c_i\}_{i=1}^{\infty}$ such that

$$\Theta = \sum_{i=1}^{\infty} c_i f_i = s_{\nu_0} + Ext\text{-}\sum_{m=0}^{\omega} \left[s_{\nu_{m+1}} - \nu_m\right]. \tag{5.72}$$

Proof. We will choose now rapidly increasing sequence of indices $\{c_{\nu_m}\}_{m=1}^{\infty}$ such that

$$_{k=\nu_m+1}^{\nu_{m+1}} \left\|s_{\nu_{m+1}} - s_{\nu_m}\right\|_2 \leq \frac{1}{2^{m+1}}. \tag{5.73}$$

It follows from (5.5.3) that for any $n \in \mathbb{N}$ such that $\nu_m < n < \nu_{m+1}$

$$\begin{gathered} \|\Theta - s_n\|_2 \leq \|s_n - s_{\nu_m}\|_2 + Ext\text{-}_{i=m}^{\infty} \left\|s_{\nu_{i+1}} - s_{\nu_i}\right\|_2 = \\ =_{k=\nu_m+1}^{n} \|c_k f_k\|_2 + Ext\text{-}_{i=m}^{\infty} \left[_{k=\nu_i+1}^{\nu_{i+1}} \|c_k f_k\|_2\right] \leq \frac{1}{2^{m+1}} +_{i=m}^{\infty} \frac{1}{2^{i+1}} \underset{m\to\infty}{\to} 0. \end{gathered} \tag{5.74}$$

From the Eq.(5.5.1) and (5.5.7) we obtain

$$\Theta = \sum_{i=1}^{\infty} c_i f_i = s_{\nu_0} + \sum_{m=0}^{\infty} \left[s_{\nu_{m+1}} - \nu_m\right]. \tag{5.75}$$

where both series in RHS of the Eq.(5.5.8) converges absolutely in the norm $\|\cdot\|_2$.

Let Θ_1 be the external sum

$$\Theta_1 = s_{\nu_0} + Ext\text{-}\sum_{m=0}^{\omega} \left[s_{\nu_{m+1}} - \nu_m \right]. \tag{5.76}$$

From Eq.(5.5.9) we obtain

$$\begin{gathered} \|\Theta_1 - s_n\|_2 \le \|s_n - s_{\nu_m}\|_2 + Ext\text{-}_{i=m}^{\omega} \left\| s_{\nu_{i+1}} - s_{\nu_i} \right\|_2 = \\ =_{k=\nu_m+1}^{n} \|c_k f_k\|_2 + Ext\text{-}_{i=m}^{\omega} \left[_{k=\nu_i+1}^{\nu_{i+1}} \|c_k f_k\|_2 \right] \le \frac{1}{2^{m+1}} + Ext\text{-}_{i=m}^{\omega} \frac{1}{2^{i+1}} \underset{m\to\infty}{\to} 0. \end{gathered} \tag{5.77}$$

From Eq.(2.5.9), Eq.(2.5.4) and (2.5.8) we obtain for any $n \in \mathbb{N}$ such that $\nu_m < n < \nu_{m+1}$:

$$\|\Theta_1 - s_n\|_2 \underset{m\to\infty}{\to} 0. \tag{5.78}$$

by Theorem 3.12.6 in [18], we obtain that

$$\Theta_1 = s_{\nu_0} + Ext\text{-}\sum_{m=0}^{\omega} \left[s_{\nu_{m+1}} - \nu_m \right] = s_{\nu_0} + \sum_{m=0}^{\infty} \left[s_{\nu_{m+1}} - \nu_m \right]. \tag{5.79}$$

From Eq.(5.5.8) and Eq.(5.5.12) finally we get

$$\Theta = \sum_{i=1}^{\infty} c_i f_i = s_{\nu_0} + Ext\text{-}\sum_{m=0}^{\omega} \left[s_{\nu_{m+1}} - \nu_m \right]. \tag{5.80}$$

Proposition 5.5.2. The all basis vectors $\{\mathbf{e}_k\}_{k=1}^{\infty}$ of l_2 also are in subspace Δ_1.

Proof. It follofs from Assumption 5.5.1 that any vector Θ_2 has a representative (5.5.1) and by Proposition 5.5.1 we obtain that: $\Theta \in l_2 \implies \Theta \in \Delta_1$.

Theorem 5.5.1. If $\mathbf{A} : l_2 \to l_2$ is a bounded non-trivial linear operator on a complex space l_2, it follow that $\mathbf{A}$ has a non-trivial closed invariant subspace.

Proof. Assume that the a bounded linear operator $\mathbf{A} : l_2 \to l_2$ hasn't non trivial closed invariant subspace. It follow by Proposition 5.5.2 that $\Delta_1 = l_{2,\omega}^{\#}$ but this is a contradiction,since $\Delta_1 \subsetneqq l_{2,\omega}^{\#} = V_{\mathbf{m}}^{(1)} = \Delta_1 \oplus \Delta_2$.This contradiction finalized the proof.

6 The Invariant Subspace Problem of Hilbert Operator Algebras Over Field ${}^*\mathbb{C}_c^{\#} = {}^*\mathbb{R}_c^{\#} + \mathbf{i}\ {}^*\mathbb{R}_c^{\#}$. Positive Nonclassical Results

Definition 6.1. Any collection of linear transformations on external vector space $V_n = V_n\left[{}^*\mathbb{C}_c^{\#}\right]$ (over field ${}^*\mathbb{C}_c^{\#}$) of hyperfinite or finite dimension $\dim_{{}^*\mathbb{C}_c^{\#}}(V_n) = n \in \mathbb{N}^{\#}$ is said to be irreducible if the only linear subspaces that are invariant under all the operators in the collection are $\{0\}$ and the entire space V_n.

Theorem 6.1.[21] (Burnside's Theorem).The only irreducible algebra of linear transformations on a vector space V_n of finite dimension $n = \dim(V_n), n \in \mathbb{N}$ greater than 1 over an algebraically closed field is the algebra of all linear transformations on the vector space.

Definition 6.2. The rank of a linear transformation L on external vector space V_n is the dimension of its image, written az $\mathbf{rank}\,(L)$.

Theorem 6.2. (Generalized Burnside's Theorem). The only irreducible algebra of linear transformations on external vector space V_n of hyperfinite or finite dimension $\dim_{{}^*\mathbb{C}_c^\#}(V_n) = n \in \mathbb{N}^\#$ greater than 1 over field ${}^*\mathbb{C}_c^\# = {}^*\mathbb{R}_c^\# + \mathbf{i}\ {}^*\mathbb{R}_c^\#$ is the algebra of all linear transformations on the external vector space V_n.

Proof. Say that an algebra $\Re$ of linear transformations on external vector space V_n is transitive if $\{Ax|A \in \Re\}$ is V_n for every vector x other than 0. For every x, $\{Ax|A \in \Re\}$ is an invariant subspace for A, so an algebra of linear transformations on a space of dimension greater than 1 is transitive if and only if it is irreducible. (On a one-dimensional space, the algebra $\{0\}$ is irreducible but not transitive.

We now prove the following by induction on the dimension of V_n: If $\Re$ is a transitive algebra of linear transformations on V_n, then $\Re$ is the algebra of all linear transformations on V_n.

Let n denote the dimension of V_n. The above assertion is obviously true for $n = 1$, so we assume it holds on all spaces of dimension from 1 to $n - 1$, and suppose that $\Re$ is a transitive algebra on a space of dimension n.

First, note that $\Re$ contains a non-invertible operator F other than 0. To see this, let T be any operator in $\Re$ that is not a scalar multiple of the identity. If T is not invertible, let $F = T$. If T is invertible, let $3bb$ be any eigenvalue of T, and then let $F = 3bbT - T^2$.Since F is not invertible, the dimension of its range, FV_n, is less than n.

The set of all restrictions to FV_n of $\{FA : A \in \Re\}$ is a transitive algebra on FV_n, so the inductive hypothesis implies that this algebra contains all linear transformations on FV_n. In particular, there is an $A_0 \in \Re$ such that the restriction of FA_0 to FV_n has rank 1. Then FA_0F is a transformation of rank 1 in $\Re$.

Note that if a transitive algebra $\Re$ contains a transformation of rank 1,then it contains all transformations. For suppose that $\Re$ contains the transformation $y_0 \otimes f_0$, with y_0 in the space and f_0 in the dual space (defined by $(y_0 \otimes f_0)(x) = f_0(x)y_0$ for all $x \in V_n$).

For $A \in \Re, A(y_0 \otimes f_0) = Ay_0 \otimes f_0$, so the transitivity of $\Re$ implies that $y \otimes f_0$ is in $\Re$ for every y. It is clear that the transitivity of $\Re$ implies transitivity of the set of all operators on the dual space of V_n that are duals of operators in $\Re$.

Moreover, $(y \otimes f_0)A = y \otimes A' f_0$, where A' is the dual of A.Hence $y \otimes f$ is in $\Re$ for all y in V_n and f in the dual space; i.e., $\Re$ contains all transformations on V_n of rank 1. Since every linear transformation is a sum of transformations of rank 1, $\Re$ is the algebra of all linear transformations on V_n.

Theorem 6.3. Let H be infinite-dimensional complex separable Hilbert space. Let $\Re$ be a family of operators $\Re \subset B(H)$ and $(\Re)$ is the algebra generated by $\Re$.

Then $\mathbf{lat}\,(\Re) = \{0, H\} \iff (\Re) = B(H)$.

Proof. Immediately by Theorem 6.2 from definitions.

7 Conclusion

Though the history of infinitesimals and infinity is long and tortuous, nonstandard analysis, as a canonical formulation of the method of infinitesimals, is only about 60 years old. Hence, definitive answers for many of its methodological issues are yet to be found. In 1960, Abraham Robinson, exploiting the power of the theory of formal language reinvented the method of infinitesimals, which he called nonstandard analysis because it used nonstandard models of analysis. K. Hrbacek argue for acceptance of $BNST^{+}$ (Basic Nonstandard Set Theory plus additional Idealization axioms) [22]. $BNST^{+}$ has nontrivial consequences for standard set theory; for example, it implies existence of inner models with measurable cardinals.It has been proved in [23]-[25] that any set theory wich implies existence of inner models with measurable cardinals is inconsistent. However hyper Infinitary first-order logic ${}^{2}L^{\#}_{\infty^{\#}}$ with restricted rules of conclusions obviously can save $BNST^{+}$ from a triviality.

Competing Interests

Author has declared that no competing interests exist.

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Appendix

Appendix A. Generalized fundamental theorem of algebra

Remind that a hyperfinite polynomial in a single indeterminate x can always be written symbollicaly in the form

$$p\left(x\right) = Ext\text{-}\left(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0\right), \tag{7.1}$$

where $a_0, \ldots, a_n \in {}^*\mathbb{R}_c^{\#}$ are constants and x is the indeterminate. The word indeterminate means that x represents no particular value, although an value may be substituted for it. The mapping that associates the result of this substitution to the substituted value is a function, called a polynomial function of the hyperfinite degree
$n \in \mathbb{N}^{\#}\backslash\mathbb{N}$.

Definition 1. The external polynomial function of the hyperfinite degree $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$ is given by

$$p\left(x\right) = Ext\text{-}_{k=0}^{n} a_k x^k. \tag{7.2}$$

Theorem 1.[19].Every univariate polynomial (2) of positive degree $n \in \mathbb{N}^{\#}$ with real coefficients in field ${}^*\mathbb{R}_c^{\#}$ has at least one complex root in field $\mathbb{C}_c^{\#} = {}^*\mathbb{R}_c^{\#} + \mathbf{i}\ {}^*\mathbb{R}_c^{\#}$.

A.1.Topological proof

For topological proof by contradiction, suppose that the polynomial $p(z)$ has no roots, and consequently is never equal to 0. Think of the polynomial as a map from the complex plane $\mathbb{C}_c^{\#}$ into the complex plane $\mathbb{C}_c^{\#}$. It maps any circle $|z| = R$ into a closed loop, a curve $P(R)$. We will consider what happens to the winding number of $P(R)$ at the extremes when R is very large and when $R = 0$. When R is a sufficiently infinite large number, then the leading term z^n of $p(z)$ dominates all other terms combined; in other words,

$$|z^n| > |Ext\text{-}\left(a_{n-1}z^{n-1} + \cdots + a_0\right)|. \tag{7.3}$$

When z traverses the circle $R\times$ $(Ext\text{-}\exp\left(i3b8\right))$ ($0 \leq 3b8 \leq 23c0_{\#}$) , then
$z^n = R^n \times (Ext\text{-}\exp\left(in3b8\right))$ winds n times counter-clockwise ($0 \leq 3b8 \leq 23c0_{\#}n$) around the origin $(0, 0)$, and $P(R)$ likewise. At the other extreme, with $|z| = 0$, the curve $P(0)$is merely the single point $p(0)$, which must be nonzero because $p(z)$ is never zero.
Thus $p(0)$ must be distinct from the origin $(0, 0)$, which denotes 0 in the complex plane $\mathbb{C}_c^{\#}$.
The winding number of $P(0)$ around the origin $(0, 0)$ is thus 0. Now changing R continuously will deform the loop continuously. At some R the winding number must change.
But that can only happen if the curve $P(R)$ includes the origin $(0, 0)$ for some R. But then for some z on that circle $|z| = R$ we have $p(z) = 0$, contradicting our original assumption.
Therefore, $p(z)$ has at least one zero.

A.2.Complex #-analytic proofs

We assume by contradiction that $a = p(z_0) \neq 0$, then, expanding $p(z)$ in powers of $z - z_0$ we can write

$$p(z) = Ext\text{-}\left(a + c_k(z - z_0)^k + c_{k+1}(z - z_0)^{k+1} + \cdots + c_n(z - z_0)^n\right). \tag{7.4}$$

Here, the c_j are simply the coefficients of the polynomial $z \to p(z+z_0)$, and we let k be the index of the first coefficient following the constant term that is non-zero. But now we see that for z sufficiently close to z_0 this has behaviour asymptotically similar to the simpler polynomial $q(z) = a + c_k(z-z_0)^k$ in the sense that (as is easy to check) the function $\left|\frac{p(z)-q(z)}{(z-z_0)^{k+1}}\right|$ is bounded by some positive constant $M \in \mathbb{R}_c^{\#}$ in some neighborhood of z_0. Therefore, if we define $3b8_0 = (\arg 2061(a) + 3c0_{\#} - \arg 2061(c_k))$ and let $z = z_0 + r \times (Ext\text{-}\exp(i3b8_0))$, then for any sufficiently small positive number $r \in \mathbb{R}_c^{\#}$, since the bound M mentioned above holds and using the triangle inequality we see that

$$\begin{gathered} |p(z)| \leq |q(z)| + r^{k+1}\left|\frac{p(z)-q(z)}{r^{k+1}}\right| \leq \\ |a + (-1)c_k r^k\,|(Ext\text{-}\exp[i(arg(a) - arg(c_k))])| + Mr^{k+1} = \\ = |a| - |c_k|r^k + Mr^{k+1}. \end{gathered} \tag{7.5}$$

When r is sufficiently close to 0 this upper bound for $|p(z)|$ is strictly smaller than $|a|$, in contradiction to the definition of z_0. (Geometrically, we have found an explicit direction $3b8_0$ such that if one approaches z_0 from that direction one can obtain values $p(z)$ smaller in absolute value than $|p(z_0)|$.)

A.3.Proof by generalized Liouville's theorem

Another analytic proof can be obtained along this line of thought observing that, since $|p(z)| > |p(0)|$ outside D,the minimum of $|p(z)|$ on the whole complex plane is achieved at z_0. If $|p(z_0)| > 0$, then $1/p(z)$ is a bounded #-holomorphic function in the entire complex plane since, for each complex number $z, |1/p(z)| \leq |1/p(z_0)|$. Applying generalized Liouville's theorem [4], which states that a bounded entire function must be constant, this would imply that $1/p(z)$ is constant and therefore that $p(z)$ is constant. This gives a contradiction, and hence $p(z_0) = 0$.

A.4. Proof by the argument principle

Yet another analytic proof uses the argument principle. Let R be a positive hyperreal number large enough so that every root of $p(z)$ has absolute value smaller than R, such a number must exist because every non-constant polynomial function of degree $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$ has at most n zeros. For each $r > R$, consider the number

$$\frac{1}{23c0_{\#}i}{}_{c(r)}\frac{p'(z)}{p(z)}d^{\#}z, \tag{7.6}$$

where $c(r)$ is the circle centered at 0 with radius r oriented counterclockwise; then the argument principle says that this number is the number N of zeros of $p(z)$ in the open ball centered at 0 with radius r, which, since $r > R$, is the total number of zeros of $p(z)$. On the other hand, the integral of n/z along $c(r)$ divided by $23c0_{\#}i$ is equal to n. But the difference between the two numbers is

$$\frac{1}{23c0_{\#}i}{}_{c(r)}\left(\frac{p'(z)}{p(z)} - \frac{n}{z}\right)d^{\#}z = \frac{1}{23c0_{\#}i}{}_{c(r)}\frac{zp'(z) - np(z)}{zp(z)}d^{\#}z. \tag{7.7}$$

The numerator of the rational expression being integrated has degree at most $n - 1$ and the degree of the denominator is $n + 1$. Therefore, the number above tends to 0 as $r \to +\infty^{\#}$. But the number is also equal to $N - n$ and so $N = n$.

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