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The Testing Procedure for the Overall Lifetime Performance Index of Rayleigh Products in Multiple Production Lines Based on the Progressive Type I Interval Censored Sample

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Abstract: The lifetime performance index is a process capability index that is commonly used for the evaluation of the durability of products in life testing and reliability analysis. In the context of multiple production lines, we introduce an overall lifetime performance index and explore the relationship between this comprehensive index and individual lifetime performance indices. For products with lifespans following the Rayleigh distribution in the i th production line, we delve into the maximum likelihood estimator and asymptotic distribution to derive both the individual and overall lifetime performance indices. By establishing a predetermined target for the overall lifetime performance index, we can determine the corresponding target for each individual lifetime performance index. The testing algorithmic procedure is proposed to ascertain whether the overall lifetime performance index has reached its target value based on the maximum likelihood estimator, accompanied by figures illustrating the analysis of test power. We found that there is a monotonic relationship for the test power with various structures of parameters. Finally, a practical illustration with one numeral example is presented to demonstrate how the testing procedure is employed to evaluate the capabilities of multiple production lines.



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Keywords: Rayleigh distribution; multiple production lines; progressive type I interval censoring; maximum likelihood estimator; lifetime performance index; testing procedure

1. Introduction

Analyzing process capability is a crucial aspect in statistical quality control applications, providing insights into whether a product aligns with the specified quality requirements. The process capability index (PCI) serves as a tool to quantify this capability by assessing the extent to which a quality characteristic meets the specified specification limit. Several widely employed process capability indices have been proposed in the literature, such as C_p , C_{pk} , C_{pm} , and C_{pmk} (see Juran [1], Kane [2], Hsiang and Taguchi [3], and Pearn et al. [4]). These indices were designed for bilateral specifications. For unilateral specification, Montgomery [5] proposed a process capability index $C_L = \frac{\mu-L}{\sigma}$ for the larger-the-better quality characteristic, where μ , σ , and L are the process mean, standard deviation, and the lower specification limit. This index is well known as the lifetime performance index in life testing and reliability analysis. The primary emphasis of this investigation is the evaluation of the lifetime performance index C_L within the context of unilateral tolerance in multiple production lines. Unlike many process capability indices assuming a normal distribution, product lifetimes often follow distributions like gamma, exponential, Rayleigh, or other lifetime distributions. In this paper, we assume a Rayleigh distribution for the lifetime of products in each production line utilizing the lifetime performance index C_L due to the preference for longer product lifetimes. For products that are produced in a single production line, Tong et al. [6] developed a uniformly minimum variance unbiased

estimator (UMVUE) for complete samples under the assumption of a one-parameter exponential distribution. In practice, many experimenters are unable to collect complete samples and can only observe incomplete ones, such as progressive censored samples. Previous studies (like those by Aggarwala [7], Balakrishnan and Aggarwala [8], Hong et al. [9], Wu et al. [10], Sanjel and Balakrishnan [11], and Lee et al. [12]) focus on drawing conclusions from data that have been censored. Additionally, for step-stress accelerated life testing data, Lee et al. [13] evaluated the lifetime performance index. Wu et al. [14] proposed a testing procedure for product lifetimes in a single production line based on progressive type I interval censored samples for Rayleigh distribution. Furthermore, Wu et al. [15] implemented an experimental design for Rayleigh products under the condition of achieving a given test power or minimizing the total experimental cost for a specific cost structure. Later on, Wu et al. [16] conducted an experimental design for the lifetime performance Index of Weibull products based on the progressive type I interval censored sample. Wu and Song [17] investigated the sampling plan for progressive type I interval censoring on the lifetime performance index of the Chen lifetime distribution. These studies are based on one production line. By extending from one production line to multiple production lines, we will define an overall lifetime performance index and explore the relationship between the overall lifetime performance index and all individual lifetime performance indices. We will develop a testing procedure to see if the overall lifetime performance index reaches the target value through the use of the maximum likelihood estimator and its asymptotic normal distribution for a given level of significance. Our aim for this research is to propose an overall lifetime performance index and design a testing procedure for this index to determine if the whole production process is capable with multiple production lines. This is a study that extends from analyzing one single production line to multiple production lines. The motivation for this study stems from the fact that many manufacturing processes consist of multiple production lines.

The progressive type I interval censoring is delineated as follows: Initially, a set of products of size n undergoes a life test starting at time 0. The inspection times are predetermined and denoted as (t_1, \dots, t_m) , where t_m is the scheduled termination time of the experiment. At each inspection time t_i , the number of observed failures X_i is recorded during the interval (t_{i-1}, t_i) , and then R_i products are removed from the remaining products. This process continues until the experiment concludes at time t_m , where the number of failures X_m is observed during the interval (t_{m-1}, t_m) , and all of the remaining items are removed and lead to the termination of the life test.

Our primary objective is to formulate a hypothesis testing procedure for the overall lifetime performance index by using the maximum likelihood estimator as the test statistic based on the progressive type I interval censored sample collected from multiple production lines, assuming a Rayleigh distribution. The shape of the Rayleigh distribution is skewed to the right, and it is an asymmetric probability distribution. Therefore, this research applies the asymmetric probability distribution model to the field of lifetime performance assessment in quality control.

The subsequent sections of this paper are structured as follows: In Section 2, the overall lifetime performance index is defined for products produced in multiple production lines. The relationship between the overall lifetime performance index and all individual lifetime performance indices is also explored. In Section 3.1, we delve into the study of the maximum likelihood estimator (MLE) for the overall lifetime performance index and all individual lifetime performance indices and explore their asymptotic normal distributional properties. This is accomplished based on the progressive type I interval censored samples from multiple production lines, assuming a Rayleigh distribution. In Section 3.2, we present a testing algorithmic procedure designed for the overall lifetime performance index under a given level of significance, which equivalently tests all individual lifetime performance indices. With the aim of illustrating our proposed testing procedure, one numerical example is provided to illustrate the application of the proposed testing procedure. Finally, Section 4 encapsulates the paper with concluding remarks.

2. The Overall Lifetime Performance Index and the Conforming Rate

We consider products produced in d production lines. Suppose that the lifetime U_i of products in the i th production line follows a Rayleigh distribution with the probability density function (pdf), cumulative distribution function (cdf), and hazard function, as follows:

$$f(u) = \frac{u}{\lambda_i^2} e^{-\frac{u^2}{2\lambda_i^2}}, u \geq 0, \lambda_i > 0, \quad (1)$$

$$F(u) = 1 - e^{-\frac{u^2}{2\lambda_i^2}}, u \geq 0, \lambda_i > 0 \quad (2)$$

and

$$h(u) = \frac{f(u)}{1 - F(u)} = \frac{u}{\lambda_i^2}, \quad (3)$$

where λ_i is the scale parameter, $i = 1, \dots, d$. The larger-the-better type process capability index proposed in Montgomery [5] is defined as

$$C_L = \frac{\mu - L}{\sigma}, \quad (4)$$

where μ represents the process mean, σ represents the process standard deviation, and L denotes the specified lower specification limit. This index is also called the lifetime performance index (LPI) for products. Consider the transformation of $Y_i = U_i^2$; then, the new lifetime Y_i has a one-parameter exponential distribution with pdf, cdf, and hazard functions as

$$f(y) = \frac{1}{2\lambda_i^2} e^{-\frac{y}{2\lambda_i^2}}, y \geq 0, \lambda_i > 0, \quad (5)$$

$$F(y) = 1 - e^{-\frac{y}{2\lambda_i^2}}, y \geq 0, \lambda_i > 0 \quad (6)$$

and

$$h(y) = \frac{f(y)}{1 - F(y)} = \frac{1}{2\lambda_i^2} \quad (7)$$

The mean and standard deviation of the new lifetime are $\mu_i = E(Y_i) = 2\lambda_i^2$, $\sigma_i = \sqrt{\text{Var}(Y_i)} = 2\lambda_i^2$, $i = 1, \dots, d$. The lifetime performance index C_{L_i} for the i th production line is reduced to

$$C_{L_i} = \frac{\mu - L_i}{\sigma} = \frac{2\lambda_i^2 - L_i}{2\lambda_i^2} = 1 - \frac{L_i}{2\lambda_i^2}, i = 1, \dots, d. \quad (8)$$

It can be observed that the lifetime performance index C_{L_i} can precisely evaluate the long-term performance of products, since the smaller the hazard function, the larger the lifetime performance index C_{L_i} .

The i th production line's conformity rate is defined as the probability that the product's lifetime will exceed the lower specification limit L_i , and it is obtained as

$$P_{r_i} = P(Y_i \geq L_i) = \int_{L_i}^{\infty} \frac{1}{2\lambda_i^2} e^{-\frac{y}{2\lambda_i^2}} dy = e^{-\frac{L_i}{2\lambda_i^2}} = e^{C_{L_i} - 1}, -\infty < C_{L_i} < 1. \quad (9)$$

Apparently, the conforming rate for the i th production line increases when the lifetime performance index C_{L_i} increases. For example, if the user wishes P_{r_i} to be over 0.860708, then they can determine that the value of C_{L_i} must be 0.95 from Equation (9).

We assume that the lifetimes of products in d production lines are independent. The overall conforming rate P_r for d production lines is computed as

$$P_r = P(Y_i \geq L_i, i = 1, \dots, d) \\ = \exp\left\{-\sum_{i=1}^d \frac{L_i}{2\lambda_i^2}\right\} = \exp\left\{\sum_{i=1}^d C_{L_i} - d\right\}, -\infty < C_{L_i} < 1.$$

We define the overall lifetime performance index C_T as $P_r = \exp\left\{\sum_{i=1}^d C_{L_i} - d\right\} = \exp\{C_T - 1\}$, $-\infty < C_T < 1$ so that the overall conforming rate P_r is an increasing function of the overall lifetime performance index C_T . To solve this equation, we can find the relationship between the overall lifetime performance index and all individual lifetime performance indices as

$$C_T = \sum_{i=1}^d C_{L_i} - (d - 1), -\infty < C_T < 1. \quad (10)$$

Under the reasonable consideration of $C_{L_1} = \dots = C_{L_d} = C_L$, the relationship between the individual lifetime performance index C_L and the overall lifetime performance index C_T becomes

$$C_L = \frac{C_T + d - 1}{d}, -\infty < C_T < 1. \quad (11)$$

For example, if the engineer wished the target value of the overall lifetime performance index to be $C_T = c_0$, the individual lifetime performance index C_L for each production line can be determined by $C_L = \frac{c_0 + d - 1}{d}$. For example, if the experimenter desires the target value of the overall lifetime performance index to be $C_T = c_0 = 0.95$, then the desired target value for each production line can be determined from Equation (11) as $C_L = 0.975, 0.98333, 0.9875, 0.99, 0.99166, 0.99285, 0.99375, 0.99444, \text{ or } 0.995$ for $d = 2, 3, \dots, 10$.

3. The Testing Procedure for the Overall Lifetime Performance Index

In this section, the maximum likelihood estimator and the asymmetric distribution of the overall lifetime performance index and each individual lifetime performance index are investigated in Section 3.1. To test whether the overall lifetime performance index reaches the desired target level, the testing procedure and the power analysis for various structures of parameters are given in Section 3.2. In Section 3.3, we give one numerical example to demonstrate how to apply our proposed testing procedure.

3.1. Maximum Likelihood Estimator of the Lifetime Performance Index

For the i th production line, the progressive type I interval censored sample X_{i1}, \dots, X_{im} is collected at the observation times t_1, \dots, t_m under the progressive censoring scheme of R_1, \dots, R_m . The likelihood function for the censored sample X_{i1}, \dots, X_{im} is

$$L(\lambda_i) \propto \prod_{j=1}^m (F(t_j) - F(t_{j-1}))^{x_{ij}} (1 - F(t_j))^{R_{ij}} \propto \prod_{j=1}^m \left(1 - e^{-\frac{t_j^2 - t_{j-1}^2}{2\lambda_i^2}}\right)^{x_{ij}} e^{-\frac{t_{j-1}^2 x_{ij} + t_j^2 R_{ij}}{2\lambda_i^2}}. \quad (12)$$

The log-likelihood function is

$$\ln L(\lambda_i) \propto \sum_{j=1}^m \{x_{ij} \ln[1 - \exp(-A_{ij})] - B_{ij}\}, \quad (13)$$

where $A_{ij} = \frac{t_j^2 - t_{j-1}^2}{2\lambda_i^2}$, $B_{ij} = \frac{t_{j-1}^2 x_{ij} + t_j^2 R_{ij}}{2\lambda_i^2}$, $i = 1, \dots, d$, $j = 1, \dots, m$. When taking the derivative for the parameter λ_i from Equation (12), the log-likelihood equation becomes

$$\frac{d}{d\lambda_i} \ln L(\lambda_i) = -\frac{1}{\lambda_i^3} \sum_{j=1}^m x_{ij} \left(\frac{t_j^2 - t_{j-1}^2}{\exp\left(\frac{t_j^2 - t_{j-1}^2}{2\lambda_i^2}\right) - 1} - (t_{j-1}^2 x_{ij} + t_j^2 R_{ij}) \right) = 0. \quad (14)$$

By numerically solving the log-likelihood Equation in (14), we can obtain the maximum likelihood estimator (MLE) of λ_i denoted by $\hat{\lambda}_i$.

In order to find the asymptotic variance of the estimator $\hat{\lambda}_i$, the Fisher's information number $I(\lambda_i) = -E\left[\frac{d^2 \ln L(\lambda_i)}{d\lambda_i^2}\right]$ needs to be computed. By taking the second derivative to the likelihood function in (12), we have

$$\begin{aligned} \frac{d^2 \ln L(\lambda_i)}{d\lambda_i^2} &= \sum_{j=1}^m \left[\frac{-X_{ij}}{\left[\exp\left(\frac{t_j^2 - t_{j-1}^2}{2\lambda_i^2}\right) - 1\right]^2} \exp\left(\frac{t_j^2 - t_{j-1}^2}{2\lambda_i^2}\right) \left(\frac{t_j^2 - t_{j-1}^2}{\lambda_i^3}\right)^2 \right] \\ &+ \frac{x_{ij}}{\exp\left(\frac{t_j^2 - t_{j-1}^2}{2\lambda_i^2}\right) - 1} \frac{3(t_j^2 - t_{j-1}^2)}{\lambda_i^4} - \frac{3(t_{j-1}^2 x_{ij} + t_j^2 R_{ij})}{\lambda_i^4} \end{aligned} \quad (15)$$

Note that

$$X_{ij} | X_{i,j-1}, \dots, X_{i1} \sim \text{Binomial}\left(n - \sum_{l=1}^{j-1} X_{il}, q_{ij}\right), \quad (16)$$

where $q_{ij} = \frac{F(t_j) - F(t_{j-1})}{1 - F(t_{j-1})} = 1 - \exp\left(-\frac{t_j^2 - t_{j-1}^2}{2\lambda_i^2}\right)$, $j = 1, \dots, m$.

Hence, we have

$$E(X_{ij} | X_{i,j-1}, \dots, X_{i1}, R_{i,j-1}, \dots, R_{i1}) = n q_{ij} \prod_{l=1}^{j-1} (1 - p_{il})(1 - q_{il}), \quad j = 1, \dots, m. \quad (17)$$

Hence, the Fisher's information number is

$$I(\lambda_i) = \frac{n}{\lambda_i^2} \sum_{j=1}^m \left\{ \frac{1 - q_{ij}}{q_{ij}} \left[4 \ln^2(1 - q_{ij}) + 6 q_{ij} \ln(1 - q_{ij}) \right] + \frac{3(t_{j-1}^2 q_{ij} + t_j^2 p_{ij}(1 - q_{ij}))}{\lambda_i^2} \right\} \prod_{l=1}^{j-1} (1 - p_{il})(1 - q_{il}), \quad (18)$$

According to the distributional property of the maximum likelihood estimator, we have $\hat{\lambda}_i \xrightarrow[n \rightarrow \infty]{d} N(\lambda_i, I^{-1}(\lambda_i))$.

For likelihood-based inferences, Guillermo et al. [18] estimated the parameters for the asymmetric beta-skew alpha-power distribution by considering the maximum likelihood method, and the Fisher information matrix was derived. For other censoring schemes, like the combined generalized progressive hybrid censoring scheme, Seong and Lee [19] found the conditional maximum likelihood estimator for the scale parameter of exponential distribution. For the confidence intervals for binomial distribution in Equation (16), Jäntschi [20] proposed three alternative confidence interval calculation methods and algorithms to improve the nominal coverage probabilities compared to the Wald asymptotic interval method. Félix et al. [21] improved the performance of several confidence intervals, including the Wald, Wilson score, and arcsine confidence intervals, by using the parametric family $\hat{p}_c = \frac{X+c}{n+2c}$ with $c \geq 0$ to estimate the parameter p , where X is the number of successes out of n trials. Rather than using the confidence intervals approach, their analysis was

conducted based on the hypothesis test approach. They identified the values of c that result in the optimal testing procedure.

Based on the invariance property of the maximum likelihood estimator, the maximum likelihood estimator of C_{L_i} is obtained as

$$\hat{C}_{L_i} = 1 - \frac{L_i}{2\hat{\lambda}_i^2}. \quad (19)$$

Using the Delta method, we can show that

$$\hat{C}_{L_i} \xrightarrow[n \rightarrow \infty]{d} N(C_{L_i}, V(\hat{C}_{L_i})) \quad (20)$$

where $\hat{V}_i(\hat{C}_{L_i}) = \frac{L_i^2 \hat{V}(\hat{\lambda}_i)}{\hat{\lambda}_i^6} = \frac{L_i^2 I^{-1}(\hat{\lambda}_i)}{\hat{\lambda}_i^6}$.

Based on the invariance property of the maximum likelihood estimator, the maximum likelihood estimator of C_T is

$$\hat{C}_T = \sum_{i=1}^d \hat{C}_{L_i} - (d - 1) \quad (21)$$

Its asymptotic distribution is $\hat{C}_T \xrightarrow[n \rightarrow \infty]{d} N\left(C_T, \sum_{i=1}^d \hat{V}(\hat{C}_{L_i})\right)$, where the variance of \hat{C}_T is $\sum_{i=1}^d V(\hat{C}_{L_i})$ and its estimate is $\sum_{i=1}^d \hat{V}(\hat{C}_{L_i}) = \sum_{i=1}^d \frac{L_i^2 I^{-1}(\hat{\lambda}_i)}{\hat{\lambda}_i^6}$.

3.2. The Testing Procedure for the Overall Lifetime Performance Index and Power Analysis

For the assessment of whether the overall lifetime performance index exceeds the desired target value c_0 , we develop a statistical testing procedure in this section. The null and alternative hypotheses are $H_0 : C_T \leq c_0$ (d production lines are not capable) vs. $H_a : C_T > c_0$ (d production lines are capable). Under the condition of $C_{L1} = \dots = C_{Ld} = C_L$, if the overall lifetime performance index is $C_T = \sum_{i=1}^k C_{L_i} - (k - 1) > c_0$, we can obtain the condition of $C_{L_i} > c_0^*$, $i = 1, \dots, d$, where the target value for each individual lifetime performance index is $c_0^* = \frac{c_0 + k - 1}{k}$. The above statistical hypothesis for testing the overall lifetime performance index is equivalently set up as follows:

$H_0 : C_{L_i} \leq c_0^*$ for some i (d production lines are not capable) vs. $H_a : C_{L_i} > c_0^*$, $i = 1, \dots, d$ (d production lines are capable). This is called an intersection–union test (IUT). In order to find the rejection region by controlling the overall error rate, we use the following theorem:

Theorem 1. To test $H_0 : C_{L_i} \leq c_0^*$ for some i vs. $H_a : C_{L_i} > c_0^*$, $i = 1, \dots, d$ with level α , the rejection region for this test is $R = \bigcap_{i=1}^k R_i$, where $R_i = \{\hat{C}_{L_i} | \hat{C}_{L_i} > C_{L_i}^0\}$, the test statistic is $\hat{C}_{L_i} = 1 - \frac{L_i}{2\hat{\lambda}_i^2}$, the critical value is $C_{L_i}^0 = 1 - \frac{L_i}{2(\lambda_{i0} + Z_{\alpha'} \sqrt{I^{-1}(\lambda_{i0})})^2}$, $i = 1, \dots, d$, and $\lambda_{i0} = \sqrt{\frac{L_i}{2(1-c_0^*)}}$. The proof of Theorem 1 is given in Appendix A.

The algorithm for conducting a testing procedure for C_T is shown below (Algorithm 1). Furthermore, the test power function $h(c_1)$ of the testing procedure at the point of $C_T = dC_L - (d - 1) = c_1 > c_0$ or $C_L = \frac{c_1 + d - 1}{d} = c_1^*$ can be obtained as follows:

$$h(c_1) = P\left(\hat{C}_{L_i} > C_{L_i}^0, i = 1, \dots, d \mid c_1^* = 1 - \frac{L_i}{2\lambda_{i1}^2}, i = 1, \dots, d\right)$$

Algorithm 1. Testing procedure for the overall lifetime performance index

Step 1: For the pre-assigned target level c_0 for the overall lifetime performance index, the required target level for each individual lifetime performance index is determined as $c_0^* = \frac{c_0+k-1}{k}$ for each production line. Then the testing hypothesis $H_0 : C_{Li} \leq c_0^*$ for some i vs.

$H_a : C_{Li} > c_0^*, i = 1, \dots, d$ are constructed.

Step 2: Under the known lower specification L_i , observe the progressive type I interval censored sample X_{i1}, \dots, X_{im} at the observation times t_1, \dots, t_m with censoring schemes of R, \dots, R_m from the Rayleigh distribution.

Step 3: Compute the value of the test statistic $\hat{C}_{Li} = 1 - \frac{L_i}{2\hat{\lambda}_i^2}$, where $\hat{\lambda}_i$ is obtained by solving Equation (14).

Step 4: For significance level of α , we can calculate the critical value

$$C_{Li}^0 = 1 - \frac{L_i}{2(\lambda_{i0} + Z_{\alpha'} \sqrt{I^{-1}(\lambda_{i0})})^2} \text{ where } \lambda_{i0} = \sqrt{\frac{L_i}{2(1-c_0^*)}}, \alpha' = \alpha^{\frac{1}{d}} \text{ and } I^{-1}(\lambda_{i0}) \text{ is defined in}$$

Equation (18).

Step 5: If $\hat{C}_{Li} > C_{Li}^0$ for all $i = 1, \dots, d$, we can concluded that the overall lifetime performance index of the products adhered the required level.

Let $\lambda_{i1} = \sqrt{\frac{L_i}{2(1-c_1^*)}}$ and $\lambda_{i0} = \sqrt{\frac{L_i}{2(1-c_0^*)}}$. Then, the power function becomes

$$\begin{aligned} h(c_1) &= P\left(1 - \frac{L_i}{2\hat{\lambda}_i^2} > 1 - \frac{L_i}{2(\lambda_{i0} + Z_{\alpha'} \sqrt{I^{-1}(\lambda_{i0})})^2}, i = 1, \dots, d\right) \\ &= P(\hat{\lambda}_i > \lambda_{i0} + Z_{\alpha'} \sqrt{I^{-1}(\lambda_{i0})}, i = 1, \dots, d) \\ &= \prod_{i=1}^d P(\hat{\lambda}_i > \lambda_{i0} + Z_{\alpha'} \sqrt{I^{-1}(\lambda_{i0})}) \\ &= \prod_{i=1}^d P\left(\frac{\hat{\lambda}_i - \lambda_{i1}}{\sqrt{I^{-1}(\lambda_{i1})}} > \frac{\lambda_{i0} - \lambda_{i1} + Z_{\alpha'} \sqrt{I^{-1}(\lambda_{i0})}}{\sqrt{I^{-1}(\lambda_{i1})}}\right) \\ &= \prod_{i=1}^d \left[1 - \Phi\left(\frac{\lambda_{i0} - \lambda_{i1} + Z_{\alpha'} \sqrt{I^{-1}(\lambda_{i0})}}{\sqrt{I^{-1}(\lambda_{i1})}}\right)\right] \end{aligned} \quad (22)$$

where $\Phi(\cdot)$ is the cdf for the standard normal distribution.

Under $L_1 = \dots = L_k = L$, the power function is reduced to

$$h(c_1) = \left[1 - \Phi\left(\frac{\lambda_0 - \lambda_1 + Z_{\alpha'} \sqrt{I^{-1}(\lambda_0)}}{\sqrt{I^{-1}(\lambda_1)}}\right)\right]^d, \quad (23)$$

where $\lambda_1 = \sqrt{\frac{L}{2(1-c_1^*)}}$ and $\lambda_0 = \sqrt{\frac{L}{2(1-c_0^*)}}$.

The powers $h(c_1)$ for testing $H_0 : C_L \leq 0.85$ are computed by Equation (23), and we use R software for programming. The powers are tabulated in Tables A1–A9 for $d = 2, 3$, and 4 with $\alpha = 0.01, 0.02$, and 0.05, respectively, for $c_1 = 0.85, 0.885, 0.9, 0.915, 0.93$, and 0.96; $m = 5, 6$, and 7; $n = 70, 90$, and 110; and $p = 0.050, 0.075$ and 0.1 under $L = 0.05$ and $T = 0.5$. The power values are showcased in Figures 1–5 using the values from Tables A1–A9 to exemplify several standard cases. We obtain the following findings: (1) In Figure 1, it can be seen that the power is an increasing function of n under $d = 3$, $\alpha = 0.02$, $m = 5$, and $p = 0.01$ (other combinations of d, m, p , and α also show the same pattern). (2) In Figure 2, it can be seen that the power increases when m increases under $d = 3$, $n = 110$, $p = 0.01$, and $\alpha = 0.02$ (other combinations of d, n, p , and α also show the same pattern). (3) In Figure 3, the power is a decreasing function of p under $d = 3$, $n = 110$, $m = 5$, and $\alpha = 0.02$ (other combinations of d, n, m , and α also have the same pattern). (4) In Figure 4, the power is an increasing function of α under $d = 3$, $n = 110$, $m = 5$, and $p = 0.01$ (other combinations of d, n, m , and p also have the same pattern). (5) In Figure 5, the power is a decreasing function of d under $n = 110$, $m = 5$, $p = 0.01$, and $\alpha = 0.02$ (other combinations of n, m, p , and α also have the same pattern). The difference of power converges when c_1 increases. (6) In Figures 1–5 the

power is an Increasing function of the value of c_1 under various combinations of $d, n, m, p,$ and α .

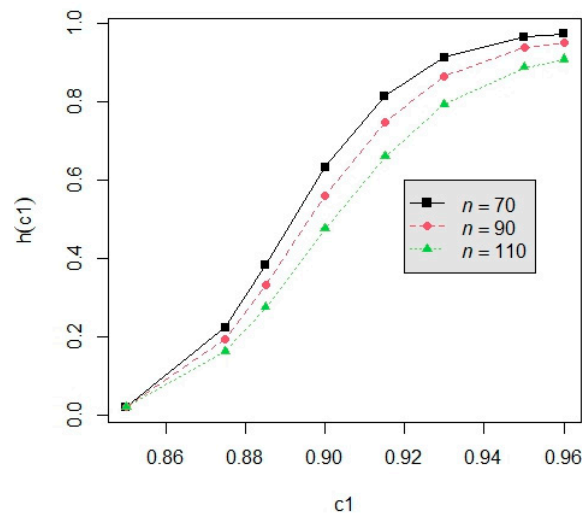


Figure 1. The power curves under $d = 3, \alpha = 0.02, m = 5, p = 0.01,$ and $n = 70, 90,$ and 110 .

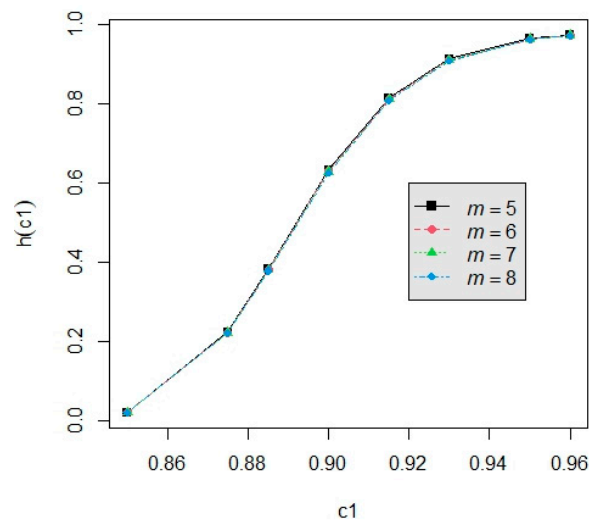


Figure 2. The power curves under $d = 3, \alpha = 0.02, n = 110, p = 0.01,$ and $m = 5, 6, 7,$ and 8 .

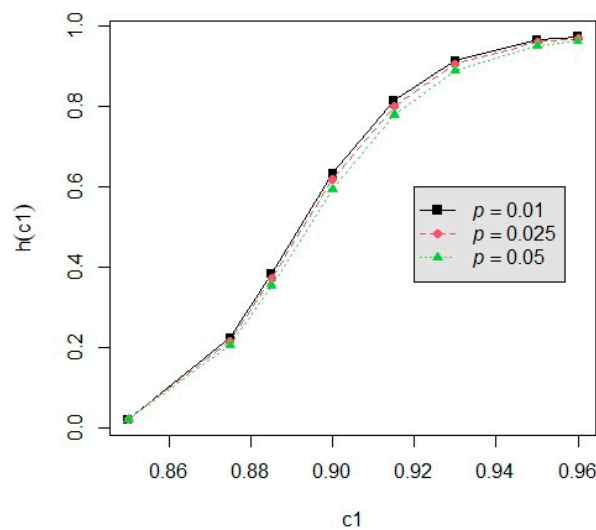


Figure 3. The power curves under $d = 3, \alpha = 0.02, n = 110, m = 5,$ and $p = 0.01, 0.025,$ and 0.05 .

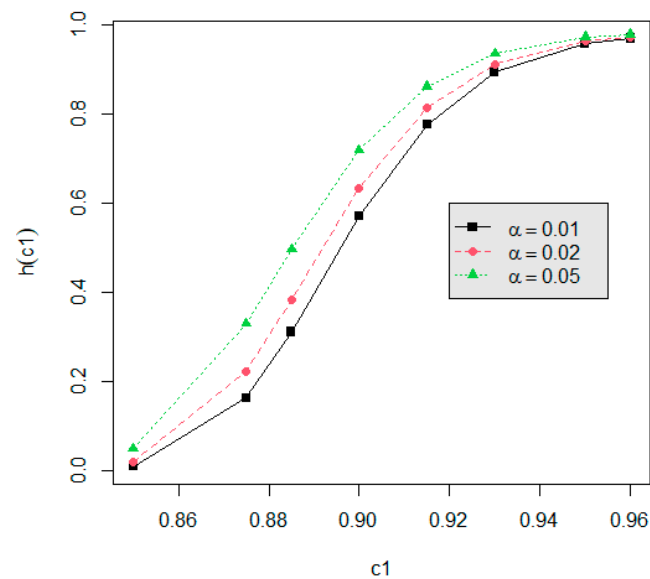


Figure 4. The power curves under $d = 3$, $n = 110$, $m = 5$, $p = 0.01$, and $\alpha = 0.01, 0.02$, and 0.05 .

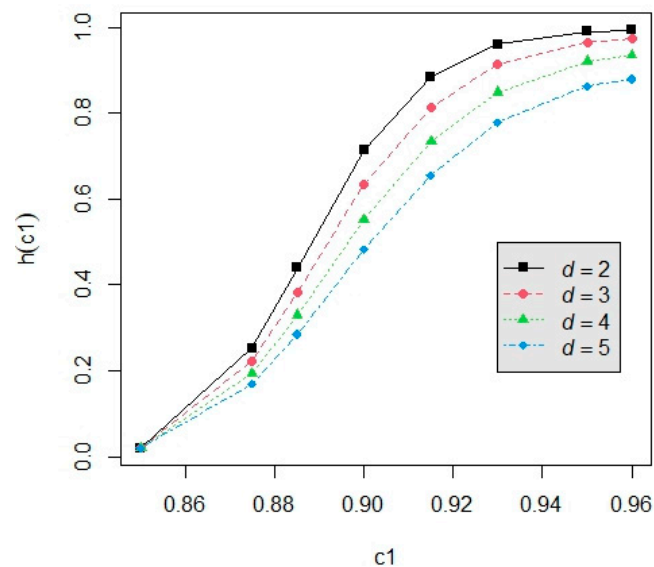


Figure 5. The power curves under $n = 30$, $m = 5$, $p = 0.01$, $\alpha = 0.02$, and $d = 1, 2, 3, 4$, and 5 .

3.3. Example

We regard the sample data in Caroni [22], consisting of the failure times of ball bearings of $n = 20$, as the lifetimes in the first production line. The data are listed as follows: 0.1788, 0.2892, 0.4152, 0.4212, 0.4560, 0.4880, 0.5184, 0.5196, 0.5412, 0.5556, 0.6780, 0.6844, 0.6864, 0.6888, 0.9312, 0.9864, 1.0512, 1.0584, 1.2804, 1.7340. We use the Gini statistic proposed in Gail and Gastwirth [23] to test the goodness of fit of the Rayleigh distribution, and the value of the test statistic does not fall into the rejection region, so these data fit the Rayleigh distribution well. We regard the sample data in Aarset [24], consisting of the failure times of $n = 50$ devices, as the lifetimes in the second production line. The data are listed as follows: 0.01, 0.02, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.2, 0.3, 0.6, 0.7, 1.1, 1.2, 1.8, 1.8, 1.8, 1.8, 1.8, 2.1, 3.2, 3.6, 4.0, 4.5, 4.5, 4.7, 5.0, 5.5, 6.0, 6.3, 6.3, 6.7, 6.7, 6.7, 6.7, 7.2, 7.5, 7.9, 8.2, 8.2, 8.3, 8.4, 8.4, 8.4, 8.5, 8.5, 8.5, 8.5, 8.5, 8.6, 8.6. The Gini test in Gail and Gastwirth [19] is used to test the goodness of fit of the Rayleigh distribution, and the p -value is $0.538 > 0.05$, so the test results support the Rayleigh distribution.

Suppose we want to test $H_0 : C_T \leq 0.85$ vs $H_1 : C_T > 0.85$. Furthermore, we create the progressive type I interval censored sample for the failure times of products from

two production lines. Let the termination time be $T = 2.0$ and let the number of inspections be $m = 5$; the equal length of inspection interval is $t = 0.1$ (thousand cycles), and the pre-specified removal percentages of the remaining survival units are given by the following: $(p_1, p_2, p_3, p_4, p_5) = (0.05, 0.05, 0.05, 0.05, 1.0)$.

Using the algorithm of the testing procedure for the overall lifetime performance index, the testing procedure is implemented as follows:

Step 1: Given the known lower specification $L_1 = L_2 = L = 0.05$ and at the observation times $(t_1, \dots, t_5) = (0.4, 0.8, 1.2, 1.6, 2.0)$, we collect the progressive type I interval censored sample, $(X_{11}, X_{12}, X_{13}, X_{14}, X_{15}) = (2, 9, 5, 1, 0)$ and $(X_{21}, X_{22}, X_{23}, X_{24}, X_{25}) = (9, 2, 1, 0, 2)$, for each production line with censoring schemes of $(R_{11}, R_{12}, R_{13}, R_{14}, R_{15}) = (1, 1, 1, 0, 0)$ and $(R_{21}, R_{22}, R_{23}, R_{24}, R_{25}) = (1, 1, 1, 1, 2)$.

Step 2: Under the required level, $c_0 = 0.85$, we can determine the required level to be 0.925 for each production line: $c_0^* = \frac{c_0 + k - 1}{k} = \frac{0.85 + 2 - 1}{2} = 0.925$. Equivalently, we can test the null hypothesis $H_0 : C_{Li} \leq 0.925$ for some i vs. $H_a : C_{Li} > 0.925, i = 1, 2$.

Step 3: For two production lines, we can obtain the maximum likelihood estimators for two production lines as $\hat{\lambda}_1 = 0.6055$ and $\hat{\lambda}_2 = 0.8789$. Then, we can compute the values of the testing statistic as $\hat{C}_{L_1} = 1 - \frac{L_1}{2\hat{\lambda}_1^2} = 1 - \frac{0.05}{2 \times (0.6055)^2} = 0.9318$ and $\hat{C}_{L_2} = 1 - \frac{L_2}{2\hat{\lambda}_2^2} = 1 - \frac{0.05}{2 \times (0.8789)^2} = 0.9676$.

Step 4: For the level of significance of $\alpha = 0.1$, we have $\alpha' = 0.1^{\frac{1}{2}} = 0.3162$ and $\lambda_{i0} = \lambda_{20} = \sqrt{\frac{L_i}{2(1-c_0^*)}} = \sqrt{\frac{0.05}{2 \times (1-0.925)}} = 0.5774$. The critical values are obtained as $C_{L_1}^0 = C_{L_2}^0 = 1 - \frac{L_i}{2(\lambda_{i0} + Z_{\alpha'} \sqrt{I^{-1}(\lambda_{i0})})^2} = 1 - \frac{0.05}{2(0.5774 + (-0.4782) \times 0.0048)^2} = 0.9156$ for two production lines.

Step 5: Since $\hat{C}_{L_1} > C_{L_1}^0$ and $\hat{C}_{L_2} > C_{L_2}^0$, we can conclude that the overall lifetime performance index has attained the required target level since all individual lifetime performance indices have reached their corresponding desired target values.

4. Conclusions

In numerous manufacturing sectors, the examination of lifetime performance indices for products is a crucial area of focus, particularly when a product's lifespan adheres to a Rayleigh distribution. For products that are produced in multiple production lines, with each one following a Rayleigh distribution for its lifetime, we introduce an overall lifetime performance index for these multiple production lines. Our investigation delves into the correlation between the overall lifetime performance index and individual lifetime performance indices. We examine the maximum likelihood estimator and asymptotic distribution for both the individual and overall lifetime performance indices in the context of progressive type I interval censored samples. To evaluate whether the overall lifetime performance index attains the desired target level, we devise a hypothesis testing procedure. This method entails testing each individual lifetime performance index using the maximum likelihood estimator as the test statistic. We explore the influences of different configurations of the sample size n , the number of inspection intervals m , the removal probability p , the level of significance α , and the number of production lines d on the test power using graphical representations and table values, with a specific focus on the number of production lines. All findings are also outlined. We found that the test power is an increasing function of n , m , and α when the other parameters are fixed. We also found that the test power is a decreasing function of p and d when the other parameters are fixed. To illustrate the application of our proposed testing algorithm for the overall lifetime performance index with two production lines, we provide one numerical example in the conclusion.

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Data Availability Statement: The data are available in a publicly accessible repository. The data presented in this study are also openly available in the studies by Caroni [22] and Aarset [24].

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A

Proof of Theorem 1. Use the maximum likelihood estimator \hat{C}_{L_i} as the test statistic. To control the overall error rate to be at most α in a multiple hypothesis testing, the probability of the type I error (false positive) is

$$\begin{aligned} P(\hat{C}_{L_i} > C_{L_i}^0, |C_{L_i} = c_0^*, i = 1, \dots, d) &= P\left(1 - \frac{L_i}{2\hat{\lambda}_i^2} > C_{L_i}^0, \left| C_{L_i} = c_0^*, i = 1, \dots, d\right.\right) \\ &= P\left(\hat{\lambda}_i > \sqrt{\frac{L_i}{2(1-C_{L_i}^0)}} \mid \lambda_i = \lambda_{i0}, i = 1, \dots, d\right), \text{ where } \lambda_{i0} = \sqrt{\frac{L_i}{2(1-c_0^*)}}. \\ &= P\left(Z > \frac{\sqrt{\frac{L_i}{2(1-C_{L_i}^0)}} - \lambda_{i0}}{\sqrt{I^{-1}(\lambda_{i0})}} \mid \lambda_i = \lambda_{i0}, i = 1, \dots, d\right) \\ &= \prod_{i=1}^d \left[1 - \Phi\left(\frac{\sqrt{\frac{L_i}{2(1-C_{L_i}^0)}} - \lambda_{i0}}{\sqrt{I^{-1}(\lambda_{i0})}}\right) \right] \leq \alpha, \end{aligned}$$

where $Z = \frac{\hat{\lambda}_i - \lambda_{i0}}{\sqrt{I^{-1}(\lambda_{i0})}} \xrightarrow[n \rightarrow \infty]{d} N(0, 1)$ and $\Phi(\cdot)$ is the cumulative distribution function for the standard normal distribution.

$$\text{Set } \left[1 - \Phi\left(\frac{\sqrt{\frac{L_i}{2(1-C_{L_i}^0)}} - \lambda_{i0}}{\sqrt{I^{-1}(\lambda_{i0})}}\right) \right] = \alpha^{\frac{1}{d}} = \alpha' \tag{A1}$$

Then, we have the probability of the type I error being at most α . Then, this test reaches the level α . From Equation (A1), we can obtain the critical value as $C_{L_i}^0 = 1 - \frac{L_i}{2(\lambda_{i0} + Z_{\alpha'} \sqrt{I^{-1}(\lambda_{i0})})^2}$, $i = 1, \dots, d$, where $Z_{\alpha'}$ represents the upper α' th percentile of a standard normal distribution. The proof is complete. \square

Table A1. The power $h(c_1)$ at $d = 2; \alpha = 0.01$.

<i>m</i>	<i>n</i>	<i>p</i>	<i>c</i> ₁					
			0.85	0.885	0.9	0.915	0.93	0.96
6	70	0.05	0.0100	0.2580	0.4975	0.7178	0.8612	0.9636
		0.075	0.0100	0.2479	0.4803	0.6997	0.8474	0.9577
		0.1	0.0100	0.2318	0.4523	0.6691	0.8229	0.9465
90	0.05	0.0100	0.0100	0.3163	0.5887	0.8041	0.9200	0.9844
		0.075	0.0100	0.3036	0.5697	0.7873	0.9094	0.9811
		0.1	0.0100	0.2835	0.5385	0.7583	0.8901	0.9745

Table A1. *Cont.*

	110	0.05	0.0100	0.3721	0.6657	0.8649	0.9540	0.9933
		0.075	0.0100	0.3573	0.6461	0.8504	0.9465	0.9915
		0.1	0.0100	0.3337	0.6135	0.8246	0.9321	0.9878
7	70	0.05	0.0100	0.2568	0.4953	0.7155	0.8595	0.9629
		0.075	0.0100	0.2447	0.4749	0.6939	0.8428	0.9557
		0.1	0.0100	0.2260	0.4421	0.6575	0.8132	0.9418
	90	0.05	0.0100	0.3147	0.5863	0.8020	0.9187	0.9840
		0.075	0.0100	0.2997	0.5638	0.7819	0.9059	0.9800
		0.1	0.0100	0.2763	0.5270	0.7471	0.8822	0.9716
	110	0.05	0.0100	0.3703	0.6632	0.8631	0.9531	0.9931
		0.075	0.0100	0.3527	0.6399	0.8457	0.9439	0.9909
		0.1	0.0100	0.3251	0.6012	0.8144	0.9260	0.9861
8	70	0.05	0.0100	0.2555	0.4931	0.7132	0.8578	0.9622
		0.075	0.0100	0.2417	0.4695	0.6881	0.8383	0.9537
		0.1	0.0100	0.2205	0.4321	0.6460	0.8035	0.9369
	90	0.05	0.0100	0.3130	0.5839	0.7999	0.9174	0.9836
		0.075	0.0100	0.2958	0.5578	0.7764	0.9023	0.9788
		0.1	0.0100	0.2693	0.5156	0.7358	0.8741	0.9685
	110	0.05	0.0100	0.3684	0.6607	0.8613	0.9522	0.9929
		0.075	0.0100	0.3482	0.6337	0.8408	0.9413	0.9902
		0.1	0.0100	0.3168	0.5891	0.8040	0.9197	0.9842

Table A2. The power $h(c_1)$ at $d = 2; \alpha = 0.02$.

m	n	p	c_1					
			0.85	0.885	0.9	0.915	0.93	0.96
6	70	0.05	0.0200	0.3276	0.5662	0.7644	0.8854	0.9685
		0.075	0.0200	0.3163	0.5493	0.7479	0.8733	0.9633
		0.1	0.0200	0.2982	0.5216	0.7199	0.8517	0.9533
90	70	0.05	0.0200	0.3910	0.6537	0.8409	0.9357	0.9868
		0.075	0.0200	0.3774	0.6358	0.8262	0.9268	0.9839
		0.1	0.0200	0.3556	0.6059	0.8006	0.9102	0.9781
110	70	0.05	0.0200	0.4499	0.7248	0.8930	0.9639	0.9944
		0.075	0.0200	0.4344	0.7070	0.8807	0.9577	0.9929
		0.1	0.0200	0.4095	0.6768	0.8586	0.9458	0.9897
7	70	0.05	0.0200	0.3262	0.5641	0.7623	0.8839	0.9679
		0.075	0.0200	0.3128	0.5440	0.7426	0.8693	0.9615
		0.1	0.0200	0.2917	0.5113	0.7091	0.8431	0.9491
90	70	0.05	0.0200	0.3893	0.6514	0.8391	0.9346	0.9864
		0.075	0.0200	0.3732	0.6301	0.8215	0.9238	0.9829
		0.1	0.0200	0.3477	0.5948	0.7906	0.9035	0.9756
110	70	0.05	0.0200	0.4479	0.7226	0.8915	0.9632	0.9942

Table A2. *Cont.*

		0.075	0.0200	0.4296	0.7013	0.8767	0.9556	0.9924
		0.1	0.0200	0.4004	0.6653	0.8498	0.9407	0.9882
8	70	0.05	0.0200	0.3248	0.5619	0.7602	0.8824	0.9672
		0.075	0.0200	0.3093	0.5387	0.7374	0.8653	0.9597
		0.1	0.0200	0.2854	0.5012	0.6984	0.8344	0.9446
	90	0.05	0.0200	0.3875	0.6492	0.8372	0.9335	0.9861
		0.075	0.0200	0.3690	0.6244	0.8167	0.9207	0.9819
		0.1	0.0200	0.3400	0.5838	0.7805	0.8965	0.9728
	110	0.05	0.0200	0.4459	0.7203	0.8900	0.9624	0.9940
		0.075	0.0200	0.4248	0.6956	0.8726	0.9534	0.9918
		0.1	0.0200	0.3915	0.6540	0.8408	0.9354	0.9866

Table A3. The power $h(c_1)$ at $d = 2; \alpha = 0.05$.

m	n	p	c_1					
			0.85	0.885	0.9	0.915	0.93	0.96
6	70	0.05	0.0500	0.4432	0.6648	0.8254	0.9158	0.9749
		0.075	0.0500	0.4309	0.6492	0.8117	0.9061	0.9706
		0.1	0.0500	0.4110	0.6233	0.7880	0.8887	0.9622
	90	0.05	0.0500	0.5098	0.7425	0.8870	0.9546	0.9897
		0.075	0.0500	0.4958	0.7269	0.8754	0.9479	0.9874
		0.1	0.0500	0.4729	0.7006	0.8549	0.9352	0.9827
	110	0.05	0.0500	0.5687	0.8024	0.9268	0.9754	0.9958
		0.075	0.0500	0.5535	0.7876	0.9176	0.9709	0.9946
		0.1	0.0500	0.5285	0.7622	0.9007	0.9621	0.9920
7	70	0.05	0.0500	0.4416	0.6629	0.8237	0.9146	0.9744
		0.075	0.0500	0.4270	0.6443	0.8073	0.9029	0.9691
		0.1	0.0500	0.4037	0.6136	0.7788	0.8817	0.9586
	90	0.05	0.0500	0.5080	0.7405	0.8855	0.9538	0.9895
		0.075	0.0500	0.4914	0.7219	0.8717	0.9456	0.9866
		0.1	0.0500	0.4645	0.6906	0.8468	0.9299	0.9807
	110	0.05	0.0500	0.5668	0.8006	0.9257	0.9749	0.9956
		0.075	0.0500	0.5487	0.7829	0.9145	0.9694	0.9942
		0.1	0.0500	0.5193	0.7524	0.8939	0.9584	0.9909
8	70	0.05	0.0500	0.4401	0.6609	0.8220	0.9134	0.9738
		0.075	0.0500	0.4232	0.6394	0.8028	0.8997	0.9675
		0.1	0.0500	0.3966	0.6040	0.7696	0.8746	0.9549
	90	0.05	0.0500	0.5062	0.7386	0.8841	0.9529	0.9892
		0.075	0.0500	0.4870	0.7170	0.8678	0.9432	0.9858
		0.1	0.0500	0.4563	0.6806	0.8386	0.9245	0.9784

Table A3. *Cont.*

110	0.05	0.0500	0.5648	0.7987	0.9245	0.9744	0.9955
	0.075	0.0500	0.5439	0.7781	0.9114	0.9678	0.9937
	0.1	0.0500	0.5102	0.7426	0.8868	0.9544	0.9896

Table A4. The power $h(c_1)$ at $d = 3; \alpha = 0.01$.

m	n	p	c_1					
			0.85	0.885	0.9	0.915	0.93	0.96
6	70	0.05	0.0100	0.2114	0.4071	0.6057	0.7572	0.8961
		0.075	0.0100	0.2026	0.3912	0.5864	0.7393	0.8842
		0.1	0.0100	0.1888	0.3657	0.5546	0.7085	0.8628
	90	0.05	0.0100	0.2613	0.4924	0.7010	0.8389	0.9435
		0.075	0.0100	0.2502	0.4741	0.6815	0.8231	0.9351
		0.1	0.0100	0.2327	0.4443	0.6487	0.7954	0.9193
	110	0.05	0.0100	0.3101	0.5679	0.7747	0.8935	0.9692
		0.075	0.0100	0.2969	0.5481	0.7563	0.8805	0.9635
		0.1	0.0100	0.2759	0.5156	0.7245	0.8570	0.9525
7	70	0.05	0.0100	0.2103	0.4051	0.6032	0.7550	0.8946
		0.075	0.0100	0.1999	0.3863	0.5803	0.7335	0.8803
		0.1	0.0100	0.1839	0.3565	0.5427	0.6967	0.8542
	90	0.05	0.0100	0.2599	0.4901	0.6985	0.8369	0.9425
		0.075	0.0100	0.2468	0.4683	0.6753	0.8180	0.9323
		0.1	0.0100	0.2264	0.4334	0.6362	0.7845	0.9128
	110	0.05	0.0100	0.3084	0.5654	0.7724	0.8919	0.9685
		0.075	0.0100	0.2928	0.5419	0.7503	0.8762	0.9616
		0.1	0.0100	0.2684	0.5036	0.7123	0.8476	0.9478
8	70	0.05	0.0100	0.2091	0.4030	0.6008	0.7527	0.8931
		0.075	0.0100	0.1972	0.3814	0.5743	0.7276	0.8763
		0.1	0.0100	0.1791	0.3475	0.5310	0.6849	0.8454
	90	0.05	0.0100	0.2585	0.4877	0.6961	0.8349	0.9414
		0.075	0.0100	0.2434	0.4627	0.6691	0.8128	0.9294
		0.1	0.0100	0.2203	0.4228	0.6239	0.7735	0.9061
	110	0.05	0.0100	0.3067	0.5629	0.7700	0.8902	0.9678
		0.075	0.0100	0.2887	0.5357	0.7443	0.8718	0.9596
		0.1	0.0100	0.2611	0.4917	0.7000	0.8379	0.9428

Table A5. The power $h(c_1)$ at $d = 3; \alpha = 0.02$.

m	n	p	c_1					
			0.85	0.885	0.9	0.915	0.93	0.96
6	70	0.05	0.0200	0.2729	0.4727	0.6574	0.7903	0.9066
		0.075	0.0200	0.2629	0.4565	0.6391	0.7738	0.8957
		0.1	0.0200	0.2470	0.4304	0.6086	0.7454	0.8758

Table A5. *Cont.*

	90	0.05	0.0200	0.3286	0.5576	0.7459	0.8637	0.9499
		0.075	0.0200	0.3163	0.5396	0.7280	0.8497	0.9423
		0.1	0.0200	0.2968	0.5101	0.6976	0.8248	0.9279
	110	0.05	0.0200	0.3814	0.6303	0.8123	0.9115	0.9730
		0.075	0.0200	0.3672	0.6114	0.7958	0.9003	0.9680
		0.1	0.0200	0.3445	0.5801	0.7673	0.8797	0.9580
7	70	0.05	0.0200	0.2716	0.4707	0.6550	0.7882	0.9053
		0.075	0.0200	0.2598	0.4515	0.6333	0.7685	0.8921
		0.1	0.0200	0.2413	0.4208	0.5972	0.7344	0.8678
	90	0.05	0.0200	0.3270	0.5553	0.7436	0.8619	0.9490
		0.075	0.0200	0.3125	0.5339	0.7223	0.8451	0.9397
		0.1	0.0200	0.2898	0.4992	0.6860	0.8150	0.9220
	110	0.05	0.0200	0.3795	0.6279	0.8102	0.9101	0.9724
		0.075	0.0200	0.3628	0.6055	0.7905	0.8965	0.9662
		0.1	0.0200	0.3363	0.5684	0.7561	0.8713	0.9538
8	70	0.05	0.0200	0.2703	0.4686	0.6527	0.7861	0.9039
		0.075	0.0200	0.2567	0.4465	0.6275	0.7631	0.8884
		0.1	0.0200	0.2357	0.4114	0.5859	0.7234	0.8597
	90	0.05	0.0200	0.3254	0.5530	0.7414	0.8602	0.9481
		0.075	0.0200	0.3088	0.5283	0.7166	0.8404	0.9371
		0.1	0.0200	0.2829	0.4885	0.6744	0.8050	0.9158
	110	0.05	0.0200	0.3777	0.6255	0.8081	0.9087	0.9718
		0.075	0.0200	0.3584	0.5995	0.7851	0.8927	0.9644
		0.1	0.0200	0.3283	0.5569	0.7450	0.8628	0.9493

Table A6. The power $h(c_1)$ at $d = 3; \alpha = 0.05$.

m	n	p	c_1					
			0.85	0.885	0.9	0.915	0.93	0.96
6	70	0.05	0.0500	0.3788	0.5717	0.7294	0.8346	0.9210
		0.075	0.0500	0.3674	0.5560	0.7131	0.8205	0.9114
		0.1	0.0500	0.3493	0.5301	0.6856	0.7959	0.8938
90	70	0.05	0.0500	0.4398	0.6518	0.8059	0.8958	0.9585
		0.075	0.0500	0.4266	0.6351	0.7908	0.8843	0.9520
		0.1	0.0500	0.4053	0.6074	0.7646	0.8637	0.9396
110	70	0.05	0.0500	0.4950	0.7171	0.8608	0.9342	0.9780
		0.075	0.0500	0.4804	0.7004	0.8474	0.9253	0.9738
		0.1	0.0500	0.4566	0.6723	0.8238	0.9088	0.9654
7	70	0.05	0.0500	0.3773	0.5697	0.7273	0.8328	0.9199
		0.075	0.0500	0.3639	0.5510	0.7079	0.8159	0.9082
		0.1	0.0500	0.3427	0.5205	0.6751	0.7863	0.8867

Table A6. *Cont.*

	90	0.05	0.0500	0.4381	0.6496	0.8040	0.8944	0.9577
		0.075	0.0500	0.4225	0.6298	0.7859	0.8805	0.9498
		0.1	0.0500	0.3975	0.5970	0.7545	0.8554	0.9344
	110	0.05	0.0500	0.4931	0.7150	0.8591	0.9331	0.9775
		0.075	0.0500	0.4758	0.6951	0.8430	0.9223	0.9723
		0.1	0.0500	0.4479	0.6616	0.8145	0.9020	0.9618
8	70	0.05	0.0500	0.3758	0.5677	0.7252	0.8310	0.9187
		0.075	0.0500	0.3604	0.5461	0.7027	0.8113	0.9050
		0.1	0.0500	0.3362	0.5111	0.6647	0.7766	0.8795
	90	0.05	0.0500	0.4364	0.6475	0.8021	0.8930	0.9569
		0.075	0.0500	0.4184	0.6246	0.7810	0.8767	0.9475
		0.1	0.0500	0.3899	0.5868	0.7443	0.8470	0.9290
	110	0.05	0.0500	0.4912	0.7129	0.8574	0.9320	0.9770
		0.075	0.0500	0.4712	0.6897	0.8386	0.9193	0.9708
		0.1	0.0500	0.4393	0.6510	0.8051	0.8951	0.9579

Table A7. The power $h(c_1)$ at $d = 4$; $\alpha = 0.01$.

m	n	p	c_1					
			0.85	0.885	0.9	0.915	0.93	0.96
6	70	0.05	0.0100	0.1756	0.3361	0.5095	0.6552	0.8072
		0.075	0.0100	0.1680	0.3218	0.4905	0.6352	0.7903
		0.1	0.0100	0.1562	0.2990	0.4595	0.6016	0.7609
	90	0.05	0.0100	0.2187	0.4141	0.6065	0.7506	0.8800
		0.075	0.0100	0.2090	0.3970	0.5860	0.7314	0.8662
		0.1	0.0100	0.1937	0.3695	0.5521	0.6984	0.8415
	110	0.05	0.0100	0.2614	0.4856	0.6863	0.8206	0.9253
		0.075	0.0100	0.2496	0.4665	0.6658	0.8033	0.9147
		0.1	0.0100	0.2311	0.4355	0.6311	0.7730	0.8950
7	70	0.05	0.0100	0.1746	0.3343	0.5070	0.6527	0.8051
		0.075	0.0100	0.1657	0.3174	0.4845	0.6288	0.7848
		0.1	0.0100	0.1520	0.2908	0.4481	0.5890	0.7495
	90	0.05	0.0100	0.2174	0.4119	0.6039	0.7482	0.8783
		0.075	0.0100	0.2060	0.3916	0.5795	0.7252	0.8617
		0.1	0.0100	0.1883	0.3595	0.5395	0.6858	0.8316
	110	0.05	0.0100	0.2598	0.4832	0.6837	0.8184	0.9240
		0.075	0.0100	0.2460	0.4605	0.6592	0.7977	0.9111
		0.1	0.0100	0.2245	0.4241	0.6180	0.7611	0.8870

Table A7. Cont.

8	70	0.05	0.0100	0.1737	0.3324	0.5046	0.6501	0.8030
		0.075	0.0100	0.1634	0.3130	0.4786	0.6224	0.7793
		0.1	0.0100	0.1479	0.2829	0.4370	0.5765	0.7380
90	70	0.05	0.0100	0.2162	0.4097	0.6012	0.7458	0.8765
		0.075	0.0100	0.2030	0.3864	0.5731	0.7190	0.8571
		0.1	0.0100	0.1830	0.3498	0.5270	0.6731	0.8215
110	70	0.05	0.0100	0.2583	0.4807	0.6811	0.8163	0.9227
		0.075	0.0100	0.2424	0.4546	0.6526	0.7920	0.9075
		0.1	0.0100	0.2181	0.4130	0.6050	0.7491	0.8786

Table A8. The power $h(c_1)$ at $d = 4; \alpha = 0.02$.

m	n	p	c_1						
			0.85	0.885	0.9	0.915	0.93	0.96	
6	70	0.05	0.0200	0.2305	0.3977	0.5629	0.6939	0.8230	
		0.075	0.0200	0.2217	0.3827	0.5443	0.6749	0.8071	
		0.1	0.0200	0.2077	0.3587	0.5137	0.6430	0.7792	
	90	70	0.05	0.0200	0.2796	0.4776	0.6560	0.7826	0.8911
			0.075	0.0200	0.2687	0.4603	0.6366	0.7649	0.8783
			0.1	0.0200	0.2513	0.4322	0.6042	0.7343	0.8552
	110	70	0.05	0.0200	0.3269	0.5486	0.7304	0.8461	0.9329
			0.075	0.0200	0.3140	0.5298	0.7115	0.8305	0.9231
			0.1	0.0200	0.2935	0.4990	0.6791	0.8030	0.9050
7	70	0.05	0.0200	0.2293	0.3958	0.5605	0.6915	0.8210	
		0.075	0.0200	0.2189	0.3780	0.5384	0.6689	0.8019	
		0.1	0.0200	0.2027	0.3499	0.5024	0.6309	0.7683	
	90	70	0.05	0.0200	0.2782	0.4754	0.6535	0.7804	0.8895
			0.075	0.0200	0.2653	0.4549	0.6304	0.7592	0.8741
			0.1	0.0200	0.2451	0.4219	0.5920	0.7225	0.8459
	110	70	0.05	0.0200	0.3252	0.5462	0.7280	0.8441	0.9317
			0.075	0.0200	0.3100	0.5239	0.7054	0.8254	0.9199
			0.1	0.0200	0.2861	0.4876	0.6668	0.7921	0.8975
8	70	0.05	0.0200	0.2282	0.3938	0.5581	0.6891	0.8190	
		0.075	0.0200	0.2163	0.3734	0.5326	0.6628	0.7967	
		0.1	0.0200	0.1979	0.3414	0.4912	0.6189	0.7573	
	90	70	0.05	0.0200	0.2768	0.4731	0.6510	0.7781	0.8879
			0.075	0.0200	0.2619	0.4495	0.6243	0.7534	0.8698
			0.1	0.0200	0.2390	0.4119	0.5799	0.7107	0.8364
	110	70	0.05	0.0200	0.3236	0.5438	0.7256	0.8422	0.9305
			0.075	0.0200	0.3061	0.5180	0.6993	0.8203	0.9165
			0.1	0.0200	0.2790	0.4764	0.6545	0.7811	0.8898

Table A9. The power $h(c_1)$ at $d = 4$; $\alpha = 0.05$.

m	n	p	c_1					
			0.85	0.885	0.9	0.915	0.93	0.96
6	70	0.05	0.0500	0.3280	0.4945	0.6407	0.7481	0.8453
		0.075	0.0500	0.3177	0.4792	0.6232	0.7311	0.8308
		0.1	0.0500	0.3013	0.4544	0.5943	0.7021	0.8052
	90	0.05	0.0500	0.3838	0.5733	0.7253	0.8260	0.9064
		0.075	0.0500	0.3715	0.5565	0.7080	0.8107	0.8951
		0.1	0.0500	0.3519	0.5289	0.6786	0.7840	0.8744
	110	0.05	0.0500	0.4351	0.6403	0.7902	0.8796	0.9432
		0.075	0.0500	0.4213	0.6228	0.7739	0.8666	0.9347
		0.1	0.0500	0.3991	0.5937	0.7457	0.8434	0.9188
7	70	0.05	0.0500	0.3267	0.4925	0.6384	0.7460	0.8435
		0.075	0.0500	0.3145	0.4744	0.6177	0.7256	0.8261
		0.1	0.0500	0.2953	0.4453	0.5834	0.6910	0.7952
	90	0.05	0.0500	0.3822	0.5711	0.7231	0.8240	0.9050
		0.075	0.0500	0.3677	0.5512	0.7024	0.8057	0.8913
		0.1	0.0500	0.3448	0.5187	0.6675	0.7735	0.8660
	110	0.05	0.0500	0.4333	0.6380	0.7882	0.8780	0.9421
		0.075	0.0500	0.4170	0.6172	0.7686	0.8624	0.9318
		0.1	0.0500	0.3909	0.5828	0.7349	0.8341	0.9122
8	70	0.05	0.0500	0.3254	0.4905	0.6362	0.7438	0.8417
		0.075	0.0500	0.3114	0.4697	0.6122	0.7201	0.8213
		0.1	0.0500	0.2895	0.4364	0.5727	0.6799	0.7850
	90	0.05	0.0500	0.3806	0.5690	0.7209	0.8221	0.9036
		0.075	0.0500	0.3640	0.5460	0.6969	0.8007	0.8875
		0.1	0.0500	0.3378	0.5087	0.6564	0.7630	0.8575
	110	0.05	0.0500	0.4315	0.6358	0.7861	0.8764	0.9411
		0.075	0.0500	0.4127	0.6117	0.7633	0.8580	0.9289
		0.1	0.0500	0.3830	0.5721	0.7240	0.8246	0.9053

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