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Existence of Monotone Positive Solution for a Fourth-Order Three-Point BVP with Sign-Changing Green's Function

Junrui Yue[1,2], Yun Zhang[1], Qingyue Bai[1]

[1]Computer Information Engineering Institute, Shanxi Technology and Business College, Taiyuan, China
[2]Shanxi Technology and Business College, Taiyuan, China
Email: yuejunrui@163.com

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Abstract

This paper is concerned with the following fourth-order three-point boundary value problem $\begin{cases} u^{(4)}(t)=f(t,u(t)), t\in[0,1], \\ u'(0)=u''(0)=u'''(\eta)=u(1)=0, \end{cases}$, where $\eta\in\left(\frac{2+\sqrt[3]{2}-\sqrt[3]{4}}{3},1\right)$, we discuss the existence of positive solutions to the above problem by applying to the fixed point theory in cones and iterative technique.

Keywords

Fourth-Order Three-Point Boundary Value Problem, Sign-Changing Green's Function, Fixed Point Index, Iterative Technique, Monotone Positive Solution, Existence

1. Introduction

Boundary value problems (BVPs for short) of fourth-order ordinary differential equations have received much attention due to their striking applications in engineering, physics, material mechanics, fluid mechanics and so on. Many authors have studied the existence of single or multiple positive solutions to some fourth-order BVPs by using Banach contraction theorem, Guo-Krasnosel'skii fixed point theorem, Leray-Schauder nonlinear alternative, fixed point index theory in cones, monotone iterative technique, the method or upper and lower solutions, degree theory, critical point theorems in conical shells and so forth see [1] [2] [3] [4] [5].

However, it is necessary to point out that, in most of the existing literature, the Green's function involved is nonnegative, which is an important condition in the study of positive solutions of BVPs.

Recently, there have been some works on positive solutions for second-order or third-order BVPs when the corresponding Green's functions are sign-changing. For example, Gao, Zhang and Ma [6] studied the following second-order periodic BVP with sign-changing Green's function

$$\begin{cases} u''(t)+\left(\dfrac{1}{2}+\varepsilon\right)^2=\lambda g(t)f(u), t\in[0,2\pi], \\ u(0)=u(2\pi), u'(0)=u'(2\pi), \end{cases}$$

where $0<\varepsilon<\dfrac{1}{2}$, $g:[0,2\pi]\to R$ is continuous, $f:[0,+\infty]\to R$ is continuous and $\lambda>0$ is a parameter. The main tool used was the Leray-Schauder fixed point theorem. In 2013 [7], by applying iterative technique, Sun and Zhao discussed the existence of monotone positive for the following third-order three-point BVP with sign-changing Green's function

$$\begin{cases} u'''(t)=f(t,u(t)), t\in[0,1], \\ u'(0)=u''(\eta)=u(1)=0. \end{cases}$$

Motivated and inspired by the above-mentioned works, in this paper, we are concerned with the following fourth-order three-point BVP with sign-changing Green's function

$$\begin{cases} u^{(4)}(t)=f(t,u(t)), t\in[0,1], \\ u'(0)=u''(0)=u'''(\eta)=u(1)=0. \end{cases} \tag{1.1}$$

We will study as follows: calculating the corresponding Green function; studying the properties of Green function; constructing the proper cone; defining the proper operator; by applying iterative technique, we obtain the existence of the positive solution for the above problem.

Theorem 1.1. *Let E be a Banach space and let K be a cone in E. Assume that* Ω_1 *and* Ω_2 *are bounded open subsets of E such that* $0\in\Omega_1$, $\bar{\Omega}_1\subset\Omega_2$, *and let* $T:K\cap(\bar{\Omega}_2\setminus\Omega_1)\to K$ *be a completely continuous operator such that either;*

1) $\|Tu\|\le\|u\|$ *for* $u\in K\cap\partial\Omega_1$ *and* $\|Tu\|\ge\|u\|$ *for* $u\in K\cap\partial\Omega_2$ *or*
2) $\|Tu\|\ge\|u\|$ *for* $u\in K\cap\partial\Omega_1$ *and* $\|Tu\|\le\|u\|$ *for* $u\in K\cap\partial\Omega_2$.

Then T has a fixed point in $K\cap(\bar{\Omega}_2\setminus\Omega_1)$.

2. Preliminaries

In this paper, we always assume that $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous and satisfies the following conditions;

(H1) for each $x\in[0,+\infty)$, the mapping $t\mapsto f(t,x)$ is decreasing;

(H2) for each $t\in[0,1]$, the mapping $x\mapsto f(t,x)$ is increasing.

Lemma 2.1. [8] *Let* $\eta \in (0,1)$. *Then for any given* $y \in X$, *the BVP*

$$\begin{cases} u^{(4)}(t) = y(t), t \in [0,1], \\ u'(0) = u''(0) = u'''(\eta) = u(1) = 0 \end{cases}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s) y(s) \mathrm{d}s, t \in [0,1],$$

where

$$G(t,s) = \frac{1}{6} \begin{cases} 3(1-t)(1+t-s)s, & s \le \min\{\eta,t\}, \\ 3s - 3s^2 + s^3 - t^3, & t \le s \le \eta, \\ (t-s)^3 - (1-s)^3, & \eta < s \le t, \\ -(1-s)^3, & s > \{\eta,t\}. \end{cases} \tag{2.1}$$

Lemma 2.2. *Green's function defined by* (*2.1*) $G(t,s)$ *has the following properties*;

1) $G(t,s) \ge 0$ *for* $(t,s) \in [0,1] \times [0,\eta]$ *and* $G(t,s) \le 0$ *for* $(t,s) \in [0,1] \times (\eta,1]$.

2) $M := \max\{|G(t,s)| : t,s \in [0,1]\} = \dfrac{\eta^3 - 3\eta^2 + 3\eta}{6} < \dfrac{1}{6}$.

Proof. Since (1) is obvious, we only prove (2). If $s \in [0,\eta]$, then we have

$$\max\{G(t,s) : t \in [0,1]\} = G(0,s) = \frac{s^3 - 3s^2 + 3s}{6} \le \frac{\eta^3 - 3\eta^2 + 3\eta}{6},$$

$$\min\{G(t,s) : t \in [0,1]\} = G(1,s) = 0;$$

If $s \in (\eta,1]$,

$$\max\{G(t,s) : t \in [0,1]\} = G(1,s) = 0,$$

$$\min\{G(t,s) : t \in [0,1]\} = G(s,s) = -\frac{(1-s)^3}{6} - \frac{(1-\eta)^3}{6},$$

which together with the $\eta \in \left(\dfrac{2 + \sqrt[3]{2} - \sqrt[3]{4}}{3}, 1\right)$ implies that

$$\max\{|G(t,s)| : (t,s) \in [0,1]\} = \max\left\{\frac{\eta^3 - 3\eta^2 + 3\eta}{6}, \frac{(1-\eta)^3}{6}\right\} = \frac{3\eta - 3\eta^2 + \eta^3}{6} < \frac{1}{6}.$$

□

Let $X = C[0,1]$ be equipped with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$ and

$$P = \{u \in X : u(t) \text{ is nonnegative and decreasing on } [0,1]\}.$$

Then it is easy to check that X is a Banach space and P is a cone in X.

Introduce an order relation $\preceq$ in X by defining $u \preceq v$ if and only if $v - u \in P$, we define an operator T on P by

$$(Tu)(t) = \int_0^1 G(t,s) f(s,u(s)) \mathrm{d}s \ge 0, u \in P, t \in [0,1].$$

Of course, if u is a fixed point of T in P, then u is a decreasing nongetative solution of BVP (1.1). Besides, because of $\eta > \frac{2+\sqrt[3]{2}-\sqrt[3]{4}}{3} > \frac{1}{3}$ and literature [8], $(Tu)(t) \ge 0$ for $u \in P$ and $(Tu)'(t) \le 0$, so $T: P \to P$. More, it follows from known textbook results, for example see proposition [4], that $T: P \to P$ is completely continuous.

In the following sections and *f* satisfies the following conditions;

(H3) there exists positive constant *r* such that $f(0,r) \le 6r$;

(H4) there exists two positive constant σ, μ and $\sigma\mu \le \frac{\sigma\eta^3}{2(1-\eta)^3}$ such that

$$\sigma(u_2-u_1) \le f(t,u_2)-f(t,u_1) \le \mu(u_2-u_1), 0 \le t \le 1, 0 \le u_1 \le u_2 \le r.$$

Note; $\sigma > 0$, $\eta \in (0,1)$, $\frac{\sigma\eta^3}{2(1-\eta)^3} > \sigma$ if and only if $\eta \in \left(\frac{2+\sqrt[3]{2}-\sqrt[3]{4}}{3}, 1\right)$.

Lemma 2.3. *Let* $P_r = \{u \in P : \|u\| \le r\}$. *Then* $T: P_r \to P_r$.

Proof. Let $u \in P_r$, then

$$0 \le u(s) \le r, s \in [0,1],$$

which together with the conditions (H1) - (H3) and (2) of Lemma 2.2, we get

$$\begin{aligned} (Tu)(t) &= \int_0^1 G(t,s) f(s,u(s)) \mathrm{d}s \\ &\le \int_0^1 |G(t,s)| f(s,u(s)) \mathrm{d}s \\ &\le \int_0^1 |G(t,s)| f(0,r) \mathrm{d}s \\ &\le 6Mr \\ &\le r, t \in [0,1], \end{aligned}$$

this indicates $\|Tu\| \le r$, in view of $Tu \in P$. Hence $T: P_r \to P_r$. □

3. Main Results

Theorem 3.1. *If we construct a iterative sequence* $v_{n+1} = Tv_n$, $n = 0,1,2,\cdots$, *where* $v_0(t) \equiv 0$, *for* $t \in [0,1]$, *then* $\{v_n\}_{n=1}^{\infty}$ *converges to* v^* *in X and* v^* *is a decreasing positive solution of BVP (1.1).*

Proof. In view of $v_0 \in P_r$ and $T: P_r \to P_r$ imply $v_n \in P_r$, $n = 1,2,\cdots$, therefore $\{v_n\}_{n=1}^{\infty}$ is a bounded set. Because of *T* is completely continuous operator, set $\{v_n\}_{n=1}^{\infty}$ is relatively compact.

By introduce prove

$$v_0 \preceq v_1 \preceq v_2 \preceq \cdots \preceq v_{n-1} \preceq v_n \preceq v_{n+1} \preceq \cdots.$$

First, it is obvious that $v_1 - v_0 = v_1 \in P$, which shows that $v_0 \preceq v_1$. Next, we assume that $v_{k-1} \preceq v_k$. Then it follows from (H2), we have

$$\begin{aligned} & v'_{k+1}(t) - v'_k(t) \\ &= (Tv_k)'(t) - (Tv_{k-1})'(t) \\ &= \int_0^t \frac{\partial G(t,s)}{\partial t}\left[f(s,v_k(s)) - f(s,v_{k-1}(s))\right]\mathrm{d}s \\ &\quad + \int_t^\eta \frac{\partial G(t,s)}{\partial t}\left[f(s,v_k(s)) - f(s,v_{k-1}(s))\right]\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&+\int_{\eta}^{1}\frac{\partial G(t,s)}{\partial t}\left[f\left(s,v_k(s)\right)-f\left(s,v_{k-1}(s)\right)\right]\mathrm{d}s\\
&=\frac{1}{2}\left\{\int_0^t\left(s^2-2st\right)\left[f\left(s,v_k(s)\right)-f\left(s,v_{k-1}(s)\right)\right]\mathrm{d}s\right.\\
&\quad\left.-\int_t^{\eta}t^2\left[f\left(s,v_k(s)\right)-f\left(s,v_{k-1}(s)\right)\right]\mathrm{d}s\right\}\\
&\le 0,\ t\in[0,\eta],
\end{aligned}
$$

It follows from (H2) and (H4) that

$$
\begin{aligned}
&v_{k+1}'(t)-v_k'(t)\\
&=\left(Tv_k\right)'(t)-\left(Tv_{k-1}\right)'(t)\\
&=\int_0^{\eta}\frac{\partial G(t,s)}{\partial t}\left[f\left(s,v_k(s)\right)-f\left(s,v_{k-1}(s)\right)\right]\mathrm{d}s\\
&\quad+\int_{\eta}^{t}\frac{\partial G(t,s)}{\partial t}\left[f\left(s,v_k(s)\right)-f\left(s,v_{k-1}(s)\right)\right]\mathrm{d}s\\
&\quad+\int_{t}^{1}\frac{\partial G(t,s)}{\partial t}\left[f\left(s,v_k(s)\right)-f\left(s,v_{k-1}(s)\right)\right]\mathrm{d}s\\
&=\frac{1}{2}\left\{\int_0^{\eta}\left(s^2-2st\right)\left[f\left(s,v_k(s)\right)-f\left(s,v_{k-1}(s)\right)\right]\mathrm{d}s\right.\\
&\quad\left.+\int_{\eta}^{t}(t-s)^2\left[f\left(s,v_k(s)\right)-f\left(s,v_{k-1}(s)\right)\right]\mathrm{d}s\right\}\\
&\le\frac{1}{2}\left\{\sigma\int_0^{\eta}\left(s^2-2st\right)\left[v_k(s)-v_{k-1}(s)\right]\mathrm{d}s+\mu\int_{\eta}^{t}\left(t-s^2\right)\left[v_k(s)-v_{k-1}(s)\right]\mathrm{d}s\right\}\\
&\le\frac{v_k(\eta)-v_{k-1}(\eta)}{2}\left[\sigma\int_0^{\eta}\left(s^2-2st\right)\mathrm{d}s+\mu\int_{\eta}^{t}\left(t-s^2\right)\mathrm{d}s\right]\\
&=\frac{v_k(\eta)-v_{k-1}(\eta)}{6}\left[\mu(t-\eta)^3+\sigma\left(-3\eta^2t+\eta^3\right)\right]\\
&\le\frac{v_k(\eta)-v_{k-1}(\eta)}{6}\left[2\mu(1-\eta)^3-\sigma\eta^3\right]\\
&\le 0,\ t\in[\eta,1].
\end{aligned}
$$

hence,

$$v_{k-1}'(t)-v_k'(t)\preceq 0,\ t\in[0,1],$$

that is

$$
\begin{aligned}
v_{k+1}(t)-v_k(t)&\ge v_{k+1}(1)-v_k(1)\\
&=\int_0^1 G(1,s)\left[f\left(s,v_k(s)\right)-f\left(s,v_{k-1}(s)\right)\right]\mathrm{d}s\\
&=0,\ t\in[0,1],
\end{aligned}
$$

which indicates that $v_k\preceq v_{k+1}$. Thus, we have shown that $v_n\preceq v_{n+1}, n=1,2,\cdots$.

Since $\{v_n\}_{n=1}^{\infty}$ is relatively compact and monotone, there exist a $v^*\in P_r$. Such that $\left\|v_n-v^*\right\|\to 0\,(n\to\infty)$ which together with the continuity of T and the fact that $v_{n+1}=Tv_n$, $n=0,1,2,\cdots$ implies that $v^*=Tv^*$. This indicates that v^* is an increasing nonnegative solution of (1.1). Moreover in view of $f(t,0)\not\equiv 0$, $t\in[0,1]$, we know that zero function is not a solution of (1.1), which shows that v^* is a positive solution of (1.1). □

Ethical Approval

We certify that this manuscript is original and has not been published and will not be submitted elsewhere for publication while being considered by boundary value problem. No data have been fabricated or manipulated (including images) to support our conclusions. And authors whose names appear on the submission have contributed sufficiently to the scientific work and therefore share collective responsibility and accountability for the results.

Authors Contributions

Yue Junrui and Zhang Yun wrote the main manuscript text and Bai Qingyue calculated all the conclusions of the article. All authors reviewed the manuscript.

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Availability of Data and Materials

Data sharing is not applicable to this article as no new data were created or analyzed is this study.

I declare the research results obtained in the research work of the authors of the papers submitted. To the best of my knowledge, this paper does not contain any research results that have been published or written by other individuals or groups, except those that have been noted and cited. Individuals and groups who have made significant contributions to the study of this paper have been clearly described in the paper.

Conflicts of Interest

The authors declare that we have no conflict of interest. This article does not contain any studies with human participants or animals performed by any of the authors. Informed consent was obtained from all individual participants included in the study.

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