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A Novel Method for Hesitant Fuzzy Multiple Attribute Group Decision Making with Incomplete Weight Information

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Original Research Article

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Abstract

Aims: The aim of this paper is to develop a novel method for dealing with multiple attribute group decision making (MAGDM) problems with hesitant fuzzy information, in which the attribute values provided by the decision makers take the form of hesitant fuzzy elements (HFEs), the information about the weights of decision makers is unknown, and the information about attribute weights is incompletely known or completely unknown.

Study Design: The developed method includes the following three stages.

Place and Duration of Study: The hesitant fuzzy set (HFS), originally proposed by Torra and Narukawa, is an efficient tool to deal with situations in which experts hesitate between several possible values to evaluate the membership degree of an element to a given set.

Methodology: The first stage establishes a quadratic programming model to determine the weights of decision makers by maximizing group consensus between the individual hesitant fuzzy decision matrices and the group hesitant fuzzy decision matrix. The second stage uses the maximizing deviation method to establish an optimization model, which derives the optimal weights of attributes under hesitant fuzzy environments. After obtaining the weights of decision makers and attributes through the above two stages, the third stage develops a hesitant fuzzy TOPSIS (HFTOPSIS) method to determine a solution with the shortest distance to the hesitant fuzzy positive ideal solution (HFPIS) and the greatest distance from the hesitant fuzzy negative ideal solution (HFNIS).

Results: A practical example is provided to illustrate the proposed method.

Conclusion: The comparison analysis with the other methods shows that the developed method has its great superiority in handling the MAGDM problems with hesitant fuzzy information.

Keywords: Hesitant Fuzzy Set (HFS), Multiple Attribute Group Decision Making (MAGDM), maximizing group consensus method, maximizing deviation method, TOPSIS.

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1 Introduction

Due to the fact that the difficulty of establishing the membership degree of an element to a given set is sometimes not because we have a margin of error (as in intuitionistic fuzzy set [1], interval-valued fuzzy set [2], or interval-valued intuitionistic fuzzy set [3]) or some possibility distribution on the possible values (as in type-2 fuzzy set [4]), but because we have some possible numerical values [5], Torra and Narukawa [5] presented a new concept of hesitant fuzzy set (HFS), in which several numerical values between 0 and 1 are simultaneously used to represent the membership degree of an element to a given set. Consequently, hesitant fuzzy set is not only an extension of fuzzy sets [6] to deal with uncertainty but also an efficient tool that can represent situations in which several membership functions for a fuzzy set are possible. Since its introduction, hesitant fuzzy set has attracted increasing attentions [7-18].

Recently, some hesitant fuzzy aggregation operators [19-24] have been developed for aggregating hesitant fuzzy information. Based on these hesitant fuzzy aggregation operators, some methods [19-24] have been developed for handling the multiple attribute decision making (MADM) or multiple attribute group decision making (MAGDM) problems with hesitant fuzzy information in which the attribute values take the form of hesitant fuzzy elements (HFEs) [20] that are expressed as a set of several possible numerical values. However, these methods need to perform some aggregation operations on the input hesitant fuzzy arguments, which have some drawbacks as follows: (1) when using these methods, the weight vectors of decision makers and attributes are given by the decision makers (DMs) in advance and therefore are more or less subjective and insufficient; (2) when using these methods, the dimensions of the aggregated hesitant fuzzy elements may increase. Especially, if the dimensions of the input hesitant fuzzy elements are a little large, then the dimensions of the aggregated hesitant fuzzy elements will be very large. Consequently, it may increase the computational complexity and cause the loss of decision information. However, in many MAGDM problems with hesitant fuzzy information, because of time pressure, lack of knowledge or data, and the decision makers' limited expertise about the problem domain, the information about the weights of decision makers are unknown, and the information about the attribute weights is incompletely known or completely unknown. In addition, the larger the computational complexity, the more time that is used to obtain the optimal alternative, the higher the decision-making costs. To overcome these drawbacks, in this paper, we develop a novel method for hesitant fuzzy MAGDM with incomplete weight information. The new model can be divided into three parts: First, we establish a quadratic programming model based on the maximizing group consensus method to objectively determine the weights of decision makers. Second, we further use the maximizing deviation method to establish an optimization model, based on which the optimal attribute weights can be objectively obtained. Finally, motivated by the TOPSIS, we develop an extended TOPSIS method to determine the optimal alternative, which includes two stages. The first stage is called the hesitant fuzzy TOPSIS (HFTOPSIS), which can be used to calculate the individual relative closeness coefficient of each alternative to the individual hesitant fuzzy positive ideal solution (HFPIS). The second stage is the standard TOPSIS, which is used to calculate the group relative-closeness coefficient of each alternative to group PIS and select the optimal one with the maximum group relative-closeness coefficient. By using several illustrative examples and comparison analysis with the existing methods, our method not only is capable of handling the hesitant fuzzy MAGDM problems in which the weight information of the attributes and decision makers is unknown or partly known, but also can reduce the computational complexity and the information loss, which always happens in the process of information aggregation. Thus, our method is much appropriate for dealing with the ambiguity and hesitancy inherent in hesitant fuzzy MAGDM problems.

To do so, the rest of this paper is organized as follows. In Section 2, we briefly recall some concepts related to hesitant fuzzy sets. Section 3 develops a novel method based on the maximizing group consensus method, the maximizing deviation method and HFTOPSIS for solving the hesitant fuzzy MAGDM problem with incomplete weight information. In Section 4, an illustrative example is provided to show the effectiveness and practicality of the developed method. A comparison analysis with the other methods shows the effectiveness and practicality of the developed methods in Section 5. Section 6 provides some concluding remarks.

2 Preliminaries

Torra and Narukawa [5,25] proposed the notion of hesitant fuzzy sets to manage the situations in which several numerical values are possible for the definition of the membership of an element to a given set.

Definition 2.1 [5], [25]. Let X be a reference set, a hesitant fuzzy set (HFS) A on X is in terms of a function $h_A(x)$ that when applied to X returns a subset of $[0,1]$.

To be easily understood, we express the HFS by a mathematical symbol:

$$A=\left\{\left\langle x,h_A(x)\right\rangle \middle| x\in X\right\} \tag{1}$$

where $h_A(x)$ is a set of some values in $[0,1]$, denoting the possible membership degrees of the element $x\in X$ to the set A. For convenience, Xia and Xu [23] called $h=h_A(x)$ a hesitant fuzzy element (HFE).

Let l_h denote the numbers of values in the HFE h. For convenience, the values in the HFE h are arranged in a descending order, i.e., $h=\left\{h^{\sigma(i)} \middle| i=1,2,\cdots,l_h\right\}$, where $h^{\sigma(i)}$ is the *i*th biggest value in h.

Example 2.1. Let $X=\{x_1,x_2,x_3\}$, $A=\{\langle x_1,\{0.7,0.5\}\rangle,\langle x_2,\{0.4,0.3,0.2\}\rangle,\langle x_3,\{0.8,0.7\}\rangle\}$, and $h=\{0.4,0.3,0.2\}$. Then, A is a HFS on X, h is a HFE, and $l_h=3$.

Given three HFEs, h, h_1, and h_2, Torra and Narukawa [5,25] defined the following operations:

(1) $h^c=\bigcup_{\gamma\in h}\{1-\gamma\}$;

(2) $h_1\cup h_2=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2}\{\gamma_1\vee\gamma_2\}$; (2)

(3) $h_1\cap h_2=\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2}\{\gamma_1\wedge\gamma_2\}$. (3)

Xia and Xu [20] defined the following comparison rules for HFEs:

Definition 2.2 [20]. For a HFE $h=\bigcup_{\gamma\in h}\{\gamma\}$, $s(h)=\frac{\sum_{\gamma\in h}\gamma}{l_h}$ is called the score function of h, where l_h is the number of elements in h. For two HFEs, h_1 and h_2, if $s(h_1)>s(h_2)$, then $h_1>h_2$; if $s(h_1)=s(h_2)$, then $h_1=h_2$.

However, in some special situations, this comparison law can not distinguish two HFEs. To overcome this drawback, we further introduce the variance function of a HFE and then develop a novel method to rank two HFEs.

Definition 2.3. For a HFE $h=\bigcup_{\gamma\in h}\{\gamma\}$, $v(h)=\frac{\sum_{\gamma\in h}|\gamma-s(h)|}{l_h}$ is referred to as the variance function of h, where $s(h)$ is the score function of h.

The relationship between the score function and the variance function is similar to the relationship between the mean and variance in statistics.

Based on the score function and the variance function, we develop a comparison law to compare any two HFEs:

Definition 2.4. Let h_1 and h_2 be any two HFEs, and let $s(h_i)$ and $v(h_i)$ ($i=1,2$) be the score functions and the variance functions of h_i ($i=1,2$), respectively. Then, the following conditions hold:

(1) If $s(h_1)>s(h_2)$, then $h_1>h_2$.
(2) If $s(h_1)=s(h_2)$, then
① if $v(h_1)<v(h_2)$, then $h_1>h_2$.
② if $v(h_1)=v(h_2)$, then $h_1=h_2$.

Example 2.2. Let $h_1=\{0.5,0.4\}$ and $h_2=\{0.6,0.3\}$ be two HFEs. Then, by Definitions 2.2 and 2.3, we have

$$s(h_1)=\frac{0.4+0.5}{2}=0.45 \qquad s(h_2)=\frac{0.3+0.6}{2}=0.45$$

$$v(h_1)=\frac{|0.4-0.45|+|0.5-0.45|}{2}=0.05 \qquad v(h_2)=\frac{|0.3-0.45|+|0.6-0.45|}{2}=0.15$$

Then, $s(h_1)=s(h_2)$ and $v(h_1)<v(h_2)$. Thus, by Definition 2.4, we can obtain that $h_1>h_2$.

Let h_1 and h_2 be two HFEs. In most cases, $l_{h_1}\neq l_{h_2}$; for convenience, let $l=\max\{l_{h_1},l_{h_2}\}$. To compare h_1 and h_2, Xu and Xia [16] suggested that we should extend the shorter HFE until the length of both HFEs was the same. The simplest way to extend the shorter HFE is to append the same value repeatedly; in principle, any value can be appended. In practice, the selection of the appended value depends primarily on the decision makers' risk preferences. To address this issue, Xu and Zhang [26] developed the following method:

Definition 2.5 [26]. Assume a HFE $h=\left\{h^{\sigma(i)} \middle| i=1,2,\cdots,l_h\right\}$, and stipulate that h^+ and h^- are the maximum and minimum values in the HFE h, respectively; then we call $\bar{h}=\eta h^+ + (1-\eta)h^-$ an extension value, where η ($0 \le \eta \le 1$) is the parameter determined by the DM according his/her risk preference.

As a result, we can add different values to the HFE using h according the DM's risk preference. If $\eta=1$, then the extension value $\bar{h}=h^+$, which shows that the DM's risk preference is risk-seeking; if $\eta=0$, then $\bar{h}=h^-$, which means that the DM's risk preference is risk-averse; if $\eta=\frac{1}{2}$, then $\bar{h}=\frac{h^+ + h^-}{2}$, which indicates that the DM's risk preference is risk-neutral. Clearly, the parameter η provided by the DM reflects his/her risk preference and affects the final decision results.

Example 2.3. Let $h_1=\{0.4,0.3,0.1\}$ and $h_2=\{0.8,0.7\}$ be two HFEs. It is clear that $l_{h_1}=3$, $l_{h_2}=2$, and $l_{h_1}>l_{h_2}$. Therefore, by Xu and Zhang's method (suppose $\eta=0$), we can extend h_2 to the following: $\bar{h}_2=\{0.8,0.7,0.7\}$.

In this paper, we assume that all of the decision makers are pessimistic (other situations can be studied similarly). Xu and Xia [27] proposed a variety of distance measures for HFEs, including a hesitant normalized Hamming distance, which is defined as follows:

$$d(h_1,h_2)=\frac{1}{l}\sum_{i=1}^{l}\left|h_1^{\sigma(i)}-h_2^{\sigma(i)}\right| \tag{4}$$

where $l=\max\left\{l_{h_1},l_{h_2}\right\}$, and $h_1^{\sigma(i)}$ and $h_2^{\sigma(i)}$ are the *i*th largest values in h_1 and h_2, respectively.

Example 2.4. Let $h_1=\{0.5,0.4,0.3\}$ and $h_2=\{0.9,0.8,0.6\}$ be two HFEs. Then, $l=3$. The hesitant normalized Hamming distance of h_1 and h_2 is computed as

$$d(h_1,h_2)=\frac{1}{3}\sum_{i=1}^{3}\left|h_1^{\sigma(i)}-h_2^{\sigma(i)}\right|=\frac{1}{3}\left(|0.9-0.5|+|0.8-0.4|+|0.6-0.3|\right)=0.3667\,.$$

3 A Novel Method for Multiple Attribute Group Decision Making With Hesitant Fuzzy Information

3.1 Problem description

First, a multiple attribute group decision making (MAGDM) problem with hesitant fuzzy information can be formulated as follows: Let $X=\{x_1,x_2,\cdots,x_m\}$ be a set of m alternatives, $C=\{c_1,c_2,\ldots,c_n\}$ be a collection of n attributes, whose weight vector is $w=(w_1,w_2,\cdots,w_n)^T$, with $w_j\in[0,1]$, $j=1,2,\cdots,n$, and $\sum_{j=1}^{n}w_j=1$, and let $D=\{d_1,d_2,\cdots,d_p\}$ is a set of p decision makers, whose weight vector is $\omega=(\omega_1,\omega_2,\cdots,\omega_p)^T$, with $\omega_k\in[0,1]$, $k=1,2,\cdots,p$, and $\sum_{k=1}^{p}\omega_k=1$. Let $A^{(k)}=\left(a_{ij}^{(k)}\right)_{m\times n}$ be a hesitant fuzzy decision matrix, where $a_{ij}^{(k)}=\left\{\left(a_{ij}^{(k)}\right)^{\sigma(t)}\middle|t=1,2,\cdots,l_{a_{ij}^{(k)}}\right\}$ is a HFE, which is a set of all of the possible values that the alternative $x_i\in X$ satisfies the attribute $c_j\in C$, given by the decision maker $d_k\in D$.

In general, there are benefit attributes (i.e., the bigger the attribute values the better) and cost attributes (i.e., the smaller the attribute values the better) in a MAGDM problem. For such cases, we need to transform the hesitant fuzzy decision matrices $A^{(k)}=\left(a_{ij}^{(k)}\right)_{m\times n}$ ($k=1,2,\cdots,p$) into the normalized hesitant fuzzy decision matrix $B^{(k)}=\left(b_{ij}^{(k)}\right)_{m\times n}$ ($k=1,2,\cdots,p$) by the following equation:

$$b_{ij}^{(k)}=\begin{cases}a_{ij}^{(k)}, & \text{for benefit attribute } c_j\\ \left(a_{ij}^{(k)}\right)^c, & \text{for cost attribute } c_j\end{cases},\quad i=1,2,\cdots,m,\ j=1,2,\cdots,n,\ k=1,2,\cdots,p \quad (5)$$

where $\left(a_{ij}^{(k)}\right)^c$ is the complement of $a_{ij}^{(k)}$, such that $\left(a_{ij}^{(k)}\right)^c=\left\{1-\left(a_{ij}^{(k)}\right)^{\sigma(t)}\middle|t=1,2,\cdots,l_{a_{ij}^{(k)}}\right\}$.

In most situations, it is noted that the numbers of the elements in different HFEs $b_{ij}^{(k)}$ of $B^{(k)}$ ($k=1,2,\cdots,p$) are different. In order to more accurately calculate the distance between these HFEs, we should extend the shorter ones until all of them have the same length. Let $l=\max\left\{l_{b_{ij}^{(k)}}\middle|i=1,2,\cdots,m,\ j=1,2,\cdots,n,\ k=1,2,\cdots,p\right\}$. By the regulation method proposed by Xu and Zhang [26], we transform the hesitant fuzzy decision matrices $B^{(k)}=\left(b_{ij}^{(k)}\right)_{m\times n}$ ($k=1,2,\cdots,p$) into the corresponding hesitant fuzzy decision matrices $H^{(k)}=\left(h_{ij}^{(k)}\right)_{m\times n}$

($k=1,2,\cdots,p$), such that $l_{h_{ij}^{(k)}}=l$ for all $i=1,2,\cdots,m$, $j=1,2,\cdots,n$, and $k=1,2,\cdots,p$.

3.2 A Quadratic Programming Model for Determining the Weights of Decision Makers

First, we aggregate the individual hesitant fuzzy decision matrices $H^{(k)}=\left(h_{ij}^{(k)}\right)_{m\times n}=\left(\left\{\left(h_{ij}^{(k)}\right)^t \middle| t=1,2,\cdots,l\right\}\right)_{m\times n}$ ($k=1,2,\cdots,p$) into the group hesitant fuzzy decision matrix $H=\left(h_{ij}\right)_{m\times n}=\left(\left\{h_{ij}^t \middle| t=1,2,\cdots,l\right\}\right)_{m\times n}$, where

$$h_{ij}=\mathop{\oplus}_{k=1}^{p}\left(\omega_k h_{ij}^{(k)}\right)=\left\{\sum_{k=1}^{p}\omega_k\left(h_{ij}^{(k)}\right)^{\sigma(t)} \middle| t=1,2,\cdots,l\right\} \tag{6}$$

In general, the smaller the deviation between the individual decision information and the group decision information, the larger the consensus between the individual decision information and the group decision information, the closer that the individual decision information is to the group decision information, the more reliable the individual decision information. Therefore, the criterion of determining the optimal weights of decision makers is to minimize the deviation measure between the individual hesitant fuzzy decision matrices and the group hesitant fuzzy decision matrix.

In the following, we consider the issue how to determine the weights of decision makers, which can be classified into two cases:

(1) If all $H^{(k)}$ ($k=1,2,\cdots,p$) are the same, i.e., $H^{(k)}=H$ ($k=1,2,\cdots,p$), then it is reasonable to assign the decision makers d_k ($k=1,2,\cdots,p$) the same weight $\frac{1}{p}$.

(2) If not all of $H^{(k)}$ ($k=1,2,\cdots,p$) are the same, i.e., there at least exist two matrices $H^{(k_1)}$ and $H^{(k_2)}$ ($k_1,k_2\in\{1,2,\cdots,p\}$) such that $H^{(k_1)}\neq H^{(k_2)}$, then we introduce the deviation variables

$$e_{ij}^{(k)}(\omega)=d\left(h_{ij}^{(k)},h_{ij}\right)=\frac{\sum_{t=1}^{l}\left|\left(h_{ij}^{(k)}\right)^{\sigma(t)}-h_{ij}^{\sigma(t)}\right|}{l}=\frac{\sum_{t=1}^{l}\left|\left(h_{ij}^{(k)}\right)^{\sigma(t)}-\sum_{q=1}^{p}\omega_q\left(h_{ij}^{(q)}\right)^{\sigma(t)}\right|}{l} \tag{7}$$

for all $i=1,2,\cdots,m$, $j=1,2,\cdots,n$, $k=1,2,\cdots,p$,

and then define the square deviations among all $H^{(k)}$ ($k=1,2,\cdots,p$) and H as below:

$$
\begin{aligned}
e(\omega) &= \frac{1}{mnpl}\sum_{k=1}^{p}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{t=1}^{l}\left(\left(h_{ij}^{(k)}\right)^{\sigma(t)} - h_{ij}^{\sigma(t)}\right)^2 \\
&= \frac{1}{mnpl}\sum_{k=1}^{p}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{t=1}^{l}\left(\left(h_{ij}^{(k)}\right)^{\sigma(t)} - \sum_{q=1}^{p}\omega_q\left(h_{ij}^{(q)}\right)^{\sigma(t)}\right)^2 \\
&= \frac{1}{mnpl}\sum_{k=1}^{p}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{t=1}^{l}\left(\sum_{q=1}^{p}\omega_q\left(\left(h_{ij}^{(k)}\right)^{\sigma(t)} - \left(h_{ij}^{(q)}\right)^{\sigma(t)}\right)\right)^2 \\
&= \frac{1}{mnpl}\sum_{k=1}^{p}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{t=1}^{l}\left(\sum_{q_1=1}^{p}\omega_{q_1}\left(\left(h_{ij}^{(k)}\right)^{\sigma(t)} - \left(h_{ij}^{(q_1)}\right)^{\sigma(t)}\right)\right)\left(\sum_{q_2=1}^{p}\omega_{q_2}\left(\left(h_{ij}^{(k)}\right)^{\sigma(t)} - \left(h_{ij}^{(q_2)}\right)^{\sigma(t)}\right)\right) \\
&= \frac{1}{mnpl}\sum_{k=1}^{p}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{t=1}^{l}\left(\sum_{q_1=1}^{p}\sum_{q_2=1}^{p}\omega_{q_1}\omega_{q_2}\left(\left(h_{ij}^{(k)}\right)^{\sigma(t)} - \left(h_{ij}^{(q_1)}\right)^{\sigma(t)}\right)\left(\left(h_{ij}^{(k)}\right)^{\sigma(t)} - \left(h_{ij}^{(q_2)}\right)^{\sigma(t)}\right)\right) \\
&= \sum_{q_1=1}^{p}\sum_{q_2=1}^{p}\omega_{q_1}\omega_{q_2}\left(\frac{1}{mnpl}\sum_{k=1}^{p}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{t=1}^{l}\left(\left(\left(h_{ij}^{(k)}\right)^{\sigma(t)} - \left(h_{ij}^{(q_1)}\right)^{\sigma(t)}\right)\left(\left(h_{ij}^{(k)}\right)^{\sigma(t)} - \left(h_{ij}^{(q_2)}\right)^{\sigma(t)}\right)\right)\right)
\end{aligned}
\tag{8}
$$

It is obvious that $e(\omega)$ is the function with decision makers' weight vector $\omega = \left(\omega_1, \omega_2, \cdots, \omega_p\right)^T$. Let $G = \left(g_{q_1 q_2}\right)_{p\times p}$ be a matrix, where

$$
g_{q_1 q_2} = \frac{1}{mnpl}\sum_{k=1}^{p}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{t=1}^{l}\left(\left(\left(h_{ij}^{(k)}\right)^{\sigma(t)} - \left(h_{ij}^{(q_1)}\right)^{\sigma(t)}\right)\left(\left(h_{ij}^{(k)}\right)^{\sigma(t)} - \left(h_{ij}^{(q_2)}\right)^{\sigma(t)}\right)\right),
$$

$q_1, q_2 = 1, 2, \cdots, p$ (9)

Thus, Eq. (8) can be rewritten as

$$
e(\omega) = \omega^T G \omega \tag{10}
$$

Therefore, based on the viewpoint of maximizing group consensus, we construct the following optimal model to determine decision makers' weights in the context of GDM:

$$
\begin{aligned}
&\min e(\omega) = \omega^T G \omega \\
&\text{s.t.}\begin{cases} \sum_{k=1}^{p}\omega_k = 1, \\ \omega_k \geq 0, \quad k = 1, 2, \cdots, p, \end{cases}
\end{aligned}
\tag{M-1}
$$

Letting $E = (1, 1, \cdots, 1)^T$, we have

$$
\begin{aligned}
&\min e(\omega) = \omega^T G \omega \\
&\text{s.t.}\begin{cases} E^T \omega = 1, \\ \omega \geq 0 \end{cases}
\end{aligned}
\tag{M-2}
$$

If we take no account of the constraint of $\omega \geq 0$ temporally, then the model (M-2) becomes

$$\min e(\omega) = \omega^T G\omega$$
$$\text{s.t.} E^T \omega = 1 \tag{M-3}$$

Theorem 3.1. Let $H^{(k)} = \left(h_{ij}^{(k)}\right)_{m\times n} = \left(\left\{\left(h_{ij}^{(k)}\right)^{\sigma(t)} \middle| t = 1,2,\cdots,l\right\}\right)_{m\times n}$ ($k = 1,2,\cdots,p$) be p hesitant fuzzy decision matrices and $H = \left(h_{ij}\right)_{m\times n} = \left(\left\{h_{ij}^t \middle| t = 1,2,\cdots,l\right\}\right)_{m\times n}$ be the group hesitant fuzzy decision matrix derived from Eq. (6). If not all of $H^{(k)}$ ($k = 1,2,\cdots,p$) are the same, then the optimal solution to the model (M-3) is

$$\omega^* = \frac{G^{-1}E}{E^T G^{-1} E} \tag{11}$$

Proof. Because not all of $H^{(k)}$ ($k = 1,2,\cdots,p$) are the same, there at least exist one matrix $H^{(k_0)}$ ($k_0 \in \{1,2,\cdots,p\}$) such that $H^{(k_0)} \neq H$. Thus, there exists $i_0 \in \{1,2,\cdots,m\}$, $j_0 \in \{1,2,\cdots,n\}$, and $t_0 \in \{1,2,\cdots,l\}$, satisfying $\left(h_{i_0 j_0}^{(k_0)}\right)^{\sigma(t_0)} \neq h_{i_0 j_0}^{\sigma(t_0)}$. Therefore, we have

$$\left(\left(h_{i_0 j_0}^{(k_0)}\right)^{\sigma(t_0)} - h_{i_0 j_0}^{\sigma(t_0)}\right)^2 > 0$$

Thus,

$$e(\omega) = \frac{1}{mnpl}\sum_{k=1}^{p}\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{t=1}^{l}\left(\left(h_{ij}^{(k)}\right)^{\sigma(t)} - h_{ij}^{\sigma(t)}\right)^2 > 0 \tag{12}$$

Obviously, according to Eq. (9), we have

$$g_{q_1 q_2} = g_{q_2 q_1}, \qquad \forall q_1, q_2 = 1,2,\cdots,p$$

As a result, $G = \left(g_{q_1 q_2}\right)_{p\times p}$ is a symmetry matrix. According to Eqs. (10) and (12), we have

$$e(\omega) = \omega^T G\omega > 0;$$

Because ω is the weight vector of experts, $\omega \neq 0$. Therefore, $G = \left(g_{q_1 q_2}\right)_{p\times p}$ is a definite matrix, and it is also a nonsingular matrix. In the following, we can derive the solution to the model (M-3) by the following procedures:

We first construct the Lagrange function:

$$L(\omega,\lambda) = \omega^T G\omega + \lambda\left(E^T\omega - 1\right) \tag{13}$$

where λ is the Lagrange multiplier.

Differentiate Eq. (13) with respect to ω and λ, and then set these partial derivatives equal to zero, then we have the following equations:

$$\begin{cases} \dfrac{\partial L(\omega,\lambda)}{\partial \omega} = 2G\omega + \lambda E = 0 \\ \dfrac{\partial L(\omega,\lambda)}{\partial \lambda} = E^T\omega - 1 = 0 \end{cases} \tag{14}$$

We can obtain the optimal solution by solving Eq. (14)

$$\omega^* = \frac{G^{-1}E}{E^T G^{-1} E}$$

Because $\dfrac{\partial^2 L(\omega,\lambda)}{\partial \omega^2} = 2G$ is a definite matrix, $e(\omega) = \omega^T G\omega$ is a strictly convex function. Consequently, $\omega^* = \dfrac{G^{-1}E}{E^T G^{-1} E}$ is the unique optimal solution to the model (M-3), which completes the proof. □

If $\omega^* = \dfrac{G^{-1}E}{E^T G^{-1} E} \geq 0$, then it is also the unique optimal solution to the model (M-2); otherwise, we can utilize the LINGO (Linear Interactive and General Optimizer) software package to solve the model (M-2).

3.3 Obtaining the Optimal Weights of Attributes by the Maximizing Deviation Method

Due to the fact that many practical GDM problems are complex and uncertain and human thinking is inherently subjective, the information about attribute weights is usually incomplete. For convenience, let Δ be a set of the known weight information [27-30], where Δ can be constructed by the following forms, for $i \neq j$:

Form 1. A weak ranking: $\{w_i \geq w_j\}$;

Form 2. A strict ranking: $\{w_i - w_j \geq \alpha_i\}$ ($\alpha_i > 0$);

Form 3. A ranking of differences: $\{w_i - w_j \geq w_k - w_l\}$, for $j \neq k \neq l$;

Form 4. A ranking with multiples: $\{w_i \geq \alpha_i w_j\}$ ($0 \leq \alpha_i \leq 1$);

Form 5. An interval form: $\{\alpha_i \leq w_i \leq \alpha_i + \varepsilon_i\}$ ($0 \leq \alpha_i \leq \alpha_i + \varepsilon_i \leq 1$).

The maximizing deviation method was proposed by Wang [31] to estimate the attribute weights in MADM problems with numerical information. According to Wang [31], if the performance values of all the alternatives have small differences under an attribute, it shows that such an attribute plays a less important role in choosing the best alternative and should be assigned a smaller weight. On the contrary, if an attribute makes the performance values of all the alternatives have obvious differences, then such an attribute plays a much important role in choosing the best alternative and should be assigned a larger weight. Especially, if all available alternatives score about equally with respect to a given attribute, then such an attribute will be judged unimportant by most decision makers and should be assigned a very small weight. Wang [31] suggests that zero weight should be assigned to the attribute of this kind.

In the following, based on the maximizing deviation method, we construct an optimization model to determine the optimal relative weights of attributes under hesitant fuzzy environments. For the attribute $c_j \in C$, the deviation of the alternative $x_i \in X$ to all the other alternatives with respect to the decision maker $d_k \in D$ can be defined as below:

$$D_{ij}^{(k)} = \sum_{q=1}^{m} d\left(h_{ij}^{(k)}, h_{qj}^{(k)}\right) = \frac{\sum_{q=1}^{m}\sum_{t=1}^{l}\left|\left(h_{ij}^{(k)}\right)^{\sigma(t)} - \left(h_{qj}^{(k)}\right)^{\sigma(t)}\right|}{l}$$

$$i = 1, 2, \cdots, m, \ j = 1, 2, \cdots, n, \ k = 1, 2, \cdots, p \qquad (15)$$

Let

$$D_j^{(k)} = \sum_{i=1}^{m} D_{ij}^{(k)} = \sum_{i=1}^{m}\sum_{q=1}^{m} d\left(h_{ij}^{(k)}, h_{qj}^{(k)}\right) = \frac{\sum_{i=1}^{m}\sum_{q=1}^{m}\sum_{t=1}^{l}\left|\left(h_{ij}^{(k)}\right)^{\sigma(t)} - \left(h_{qj}^{(k)}\right)^{\sigma(t)}\right|}{l}$$

$$j = 1, 2, \cdots, n, \ k = 1, 2, \cdots, p \qquad (16)$$

then $D_j^{(k)}$ represents the deviation value of all alternatives to other alternatives for the attribute $c_j \in C$ with respect to the decision maker $d_k \in D$.

Further, let

$$\begin{aligned} D(w) &= \sum_{k=1}^{p} \omega_k \left(\sum_{j=1}^{n} w_j D_j^{(k)} \right) \\ &= \sum_{k=1}^{p} \omega_k \left(\sum_{j=1}^{n} w_j \left(\sum_{i=1}^{m} D_{ij}^{(k)} \right) \right) \\ &= \sum_{k=1}^{p} \omega_k \left(\sum_{j=1}^{n} w_j \left(\sum_{i=1}^{m} \left(\sum_{q=1}^{m} d\left(h_{ij}^{(k)}, h_{qj}^{(k)}\right) \right) \right) \right) \\ &= \frac{\sum_{k=1}^{p} \omega_k \left(\sum_{j=1}^{n} w_j \left(\sum_{i=1}^{m}\sum_{q=1}^{m}\sum_{t=1}^{l} \left| \left(h_{ij}^{(k)}\right)^{\sigma(t)} - \left(h_{qj}^{(k)}\right)^{\sigma(t)} \right| \right) \right)}{l} \\ &= \frac{\sum_{k=1}^{p} \omega_k \left(\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{q=1}^{m}\sum_{t=1}^{l} \left| \left(h_{ij}^{(k)}\right)^{\sigma(t)} - \left(h_{qj}^{(k)}\right)^{\sigma(t)} \right| w_j \right)}{l} \end{aligned} \qquad (17)$$

then $D(w)$ represents the deviation value of all alternatives to other alternatives for all the attributes with respect to all the decision makers.

From the above analysis, we can construct a non-linear programming model to select the weight vector w by maximizing $D(w)$, as follows:

$$\max D(w)=\frac{\sum_{k=1}^{p}\omega_k\left(\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{q=1}^{m}\sum_{t=1}^{l}\left|\left(h_{ij}^{(k)}\right)^{\sigma(t)}-\left(h_{qj}^{(k)}\right)^{\sigma(t)}\right|w_j\right)}{l} \qquad \text{(M-4)}$$

$$\text{s.t.}\quad w_j\geq 0,\ \ j=1,2,\cdots,n,\quad \sum_{j=1}^{n}w_j^2=1$$

To solve this model, we construct the Lagrange function:

$$L(w,\lambda)=\frac{\sum_{k=1}^{p}\omega_k\left(\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{q=1}^{m}\sum_{t=1}^{l}\left|\left(h_{ij}^{(k)}\right)^{\sigma(t)}-\left(h_{qj}^{(k)}\right)^{\sigma(t)}\right|w_j\right)}{l}+\frac{\lambda}{2}\left(\sum_{j=1}^{n}w_j^2-1\right) \qquad (18)$$

where λ is the Lagrange multiplier.

Differentiating Eq. (18) with respect to w_j ($j=1,2,\cdots,n$) and λ, and setting these partial derivatives equal to zero, then the following set of equations is obtained:

$$\frac{\partial L}{\partial w_j}=\frac{\sum_{k=1}^{p}\sum_{i=1}^{m}\sum_{q=1}^{m}\sum_{t=1}^{l}\left|\left(h_{ij}^{(k)}\right)^{\sigma(t)}-\left(h_{qj}^{(k)}\right)^{\sigma(t)}\right|\omega_k}{l}+\lambda w_j=0 \qquad (19)$$

$$\frac{\partial L}{\partial \lambda}=\frac{1}{2}\left(\sum_{j=1}^{n}w_j^2-1\right)=0 \qquad (20)$$

It follows from Eq. (19) that

$$w_j=\frac{-\sum_{k=1}^{p}\sum_{i=1}^{m}\sum_{q=1}^{m}\sum_{t=1}^{l}\left|\left(h_{ij}^{(k)}\right)^{\sigma(t)}-\left(h_{qj}^{(k)}\right)^{\sigma(t)}\right|\omega_k}{\lambda l} \qquad (21)$$

Putting Eq. (19) into Eq. (20), we get

$$\lambda = \frac{-\sqrt{\sum_{j=1}^{n}\left(\sum_{k=1}^{p}\sum_{i=1}^{m}\sum_{q=1}^{m}\sum_{t=1}^{l}\left|\left(h_{ij}^{(k)}\right)^{\sigma(t)} - \left(h_{qj}^{(k)}\right)^{\sigma(t)}\right|\omega_k\right)^2}}{l} \tag{22}$$

Then, by combining Eqs. (21) and (22), we have

$$w_j = \frac{\sum_{k=1}^{p}\sum_{i=1}^{m}\sum_{q=1}^{m}\sum_{t=1}^{l}\left|\left(h_{ij}^{(k)}\right)^{\sigma(t)} - \left(h_{qj}^{(k)}\right)^{\sigma(t)}\right|\omega_k}{\sqrt{\sum_{j=1}^{n}\left(\sum_{k=1}^{p}\sum_{i=1}^{m}\sum_{q=1}^{m}\sum_{t=1}^{l}\left|\left(h_{ij}^{(k)}\right)^{\sigma(t)} - \left(h_{qj}^{(k)}\right)^{\sigma(t)}\right|\omega_k\right)^2}} \tag{23}$$

By normalizing w_j ($j = 1, 2, \cdots, n$), we make their sum into a unit, and get

$$w_j^* = \frac{w_j}{\sum_{j=1}^{n} w_j} = \frac{\sum_{k=1}^{p}\sum_{i=1}^{m}\sum_{q=1}^{m}\sum_{t=1}^{l}\left|\left(h_{ij}^{(k)}\right)^{\sigma(t)} - \left(h_{qj}^{(k)}\right)^{\sigma(t)}\right|\omega_k}{\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{i=1}^{m}\sum_{q=1}^{m}\sum_{t=1}^{l}\left|\left(h_{ij}^{(k)}\right)^{\sigma(t)} - \left(h_{qj}^{(k)}\right)^{\sigma(t)}\right|\omega_k} \tag{24}$$

which can be considered as the optimal weight vector of attributes.

However, it is noted that there are practical situations in which the information about the weight vector is not completely unknown but partially known. For such cases, we establish the following constrained optimization model:

$$\max D(w) = \max \frac{\sum_{k=1}^{p}\omega_k\left(\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{q=1}^{m}\sum_{t=1}^{l}\left|\left(h_{ij}^{(k)}\right)^{\sigma(t)} - \left(h_{qj}^{(k)}\right)^{\sigma(t)}\right|w_j\right)}{l} \tag{M-5}$$
$$\text{s.t.} \quad w \in \Delta, \quad w_j \geq 0, \; j = 1, 2, \cdots, n, \quad \sum_{j=1}^{n} w_j = 1$$

It is noted that the model (M-5) is a linear programming model that can be solved using the MATLAB mathematics software package. Suppose that the optimal solution to the model (M-5) is $w = \left(w_1, w_2, \cdots, w_n\right)^T$, which can be considered as the weight vector of attributes.

3.4 Extended TOPIS Method for the MAGDM with Hesitant Fuzzy Information

TOPSIS method, initially introduced by Hwang and Yoon [32], is a widely used method for dealing with MADM problems, which focuses on choosing the alternative with the shortest distance from the positive ideal solution (PIS) and the farthest distance from the negative ideal solution (NIS). In the following, based on the above analysis, we shall extend the classical TOPIS method to the MAGDM problems under hesitant fuzzy environments, in which the information about the weights of decision makers is unknown, the information about attribute weights is incompletely known or completely unknown, and the attribute values are given in the form of HFEs.

The flowchart of the extended TOPIS method is provided in Fig. 1.

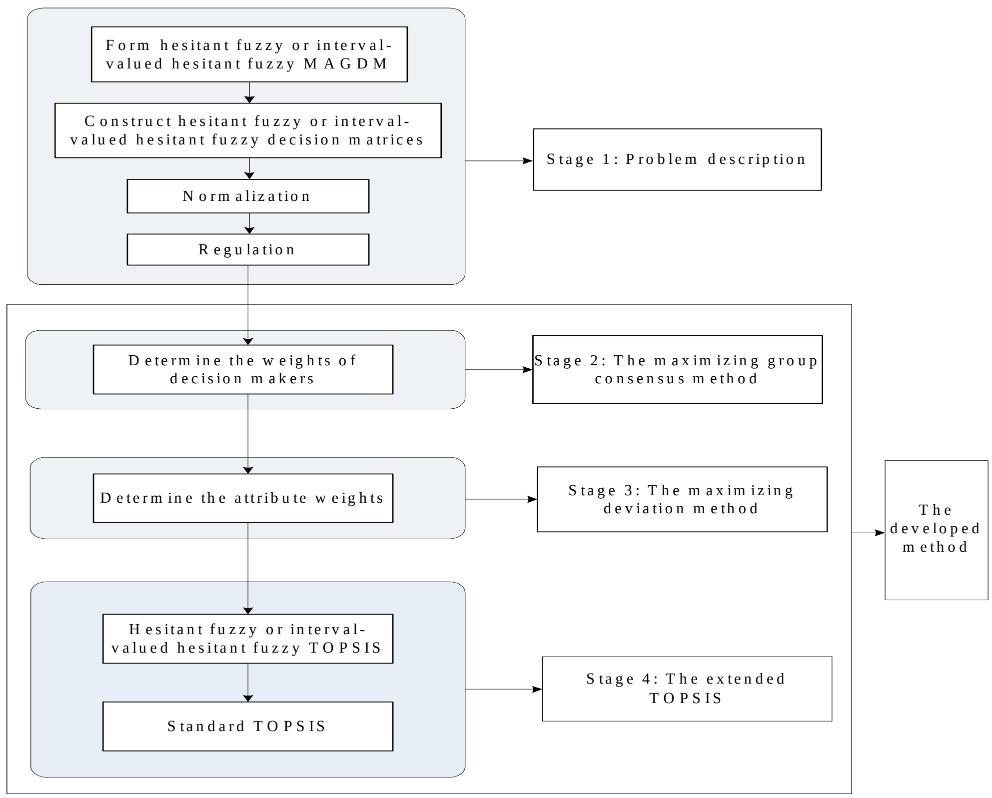

Fig. 1. The flowchart of the developed methods.

The extended method is composed of the following steps:

Step 1. For a MAGDM problem, the decision maker $d_k \in D$ constructs the hesitant fuzzy decision matrix $A^{(k)} = \left(a_{ij}^{(k)}\right)_{m\times n}$, where $a_{ij}^{(k)}$ is a HFE, given by the DM $d_k \in D$, for the alternative $x_i \in X$ with respect to the attribute $c_j \in C$. Utilize Eq. (5) to transform the hesitant fuzzy decision matrices $A^{(k)} = \left(a_{ij}^{(k)}\right)_{m\times n}$ ($k = 1, 2, \cdots, p$) into the normalized hesitant fuzzy decision matrices $H^{(k)} = \left(h_{ij}^{(k)}\right)_{m\times n}$ ($k = 1, 2, \cdots, p$).

Step 2. If the information about the weights of decision makers is unknown, then we use Eq. (11) to obtain the weights of decision makers.

Step 3. If the information about the attribute weights is completely unknown, then we use Eq. (24) to obtain the attribute weights; if the information about the attribute weights is partly known, then we solve the model (M-5) to obtain the attribute weights.

Step 4. Determine the hesitant fuzzy positive ideal solution (HFPIS) $h_{+}^{(k)} = \left\{h_{+1}^{(k)}, h_{+2}^{(k)}, \cdots, h_{+n}^{(k)}\right\}$ and the hesitant fuzzy negative ideal solution (HFNIS) $h_{-}^{(k)} = \left\{h_{-1}^{(k)}, h_{-2}^{(k)}, \cdots, h_{-n}^{(k)}\right\}$ for each decision maker d_k by the following equations:

$$h_{+j}^{(k)} = \max_i \left\{h_{ij}^{(k)}\right\} = \left\{ \max_i \left\{ \left(h_{ij}^{(k)}\right)^{\sigma(t)} \right\} \middle| t = 1, 2, \cdots, l \right\} \qquad j = 1, 2, \cdots, n \quad (25)$$

$$h_{-j}^{(k)} = \min_i \left\{h_{ij}^{(k)}\right\} = \left\{ \min_i \left\{ \left(h_{ij}^{(k)}\right)^{\sigma(t)} \right\} \middle| t = 1, 2, \cdots, l \right\} \qquad j = 1, 2, \cdots, n \quad (26)$$

Step 5. Calculate the separation measures $d_{+i}^{(k)}$ of each alternative x_i from the HFPIS $h_{+}^{(k)}$ of the decision maker d_k as:

$$d_{+i}^{(k)} = \sum_{j=1}^{n} w_j d\left(h_{ij}^{(k)}, h_{+j}^{(k)}\right) = \frac{\sum_{j=1}^{n}\sum_{t=1}^{l} w_j \left| \left(h_{ij}^{(k)}\right)^{\sigma(t)} - \left(h_{+j}^{(k)}\right)^{\sigma(t)} \right|}{l} \quad (27)$$

In a similar way, calculate the separation measures $d_{-i}^{(k)}$ of each alternative x_i from the HFNIS $h_{-}^{(k)}$ of the decision maker d_k as:

$$d_{-i}^{(k)} = \sum_{j=1}^{n} w_j d\left(h_{ij}^{(k)}, h_{-j}^{(k)}\right) = \frac{\sum_{j=1}^{n}\sum_{t=1}^{l} w_j \left| \left(h_{ij}^{(k)}\right)^{\sigma(t)} - \left(h_{-j}^{(k)}\right)^{\sigma(t)} \right|}{l} \quad (28)$$

Step 6. Calculate the relative closeness coefficient of each alternative x_i to the HFPIS $h_{+}^{(k)}$ of the decision maker d_k as:

$$C_i^{(k)} = \frac{d_{-i}^{(k)}}{d_{+i}^{(k)} + d_{-i}^{(k)}} \quad (29)$$

After calculating the $C_i^{(k)}$ for each decision maker d_k ($k = 1, 2, \cdots, p$), we then form the relative-closeness coefficient matrix as below:

$$C=\begin{pmatrix} C_1^{(1)} & C_1^{(2)} & \cdots & C_1^{(p)} \\ C_2^{(1)} & C_2^{(2)} & \cdots & C_2^{(p)} \\ \cdots & \cdots & \cdots & \cdots \\ C_m^{(1)} & C_m^{(2)} & \cdots & C_m^{(p)} \end{pmatrix}_{m\times p} \tag{30}$$

Steps 4-6 extend the standard TOPSIS to hesitant fuzzy environments and can therefore be called the hesitant fuzzy TOPSIS (HFTOPSIS). From this stage on our method continues by applying the standard TOPSIS to the relative-closeness coefficient decision matrix in order to identify the group positive ideal solution.

Step 7. Identify the group positive ideal solution (GPIS) and group negative ideal solution (GNIS), respectively as follows:

$$h_+^G=\left\{\max_i\left\{C_i^{(1)}\right\},\max_i\left\{C_i^{(2)}\right\},\cdots,\max_i\left\{C_i^{(p)}\right\}\right\} \tag{31}$$

$$h_-^G=\left\{\min_i\left\{C_i^{(1)}\right\},\min_i\left\{C_i^{(2)}\right\},\cdots,\min_i\left\{C_i^{(p)}\right\}\right\} \tag{32}$$

Step 8. Calculate the separation measures d_{+i}^G and d_{-i}^G of each alternative x_i from the group positive ideal solution h_+^G and the group negative ideal solution h_-^G, respectively, as follows:

$$d_{+i}^G=\sum_{k=1}^{p}\omega_k d\left(C_i^{(k)},\max_i\left\{C_i^{(k)}\right\}\right)=\sum_{k=1}^{p}\omega_k\left|C_i^{(k)}-\left(\max_i\left\{C_i^{(k)}\right\}\right)\right| \tag{33}$$

$$d_{-i}^G=\sum_{k=1}^{p}\omega_k d\left(C_i^{(k)},\min_i\left\{C_i^{(k)}\right\}\right)=\sum_{k=1}^{p}\omega_k\left|C_i^{(k)}-\left(\min_i\left\{C_i^{(k)}\right\}\right)\right| \tag{34}$$

Step 9. Calculate the group relative-closeness coefficient C_i^G of each alternative x_i to group positive ideal solution d_{+i}^G as:

$$C_i^G=\frac{d_{-i}^G}{d_{+i}^G+d_{-i}^G} \tag{35}$$

Step 10. Rank the alternatives x_i ($i=1,2,\cdots,m$) according to the group relative-closeness coefficients C_i^G ($i=1,2,\cdots,m$) and then select the most desirable one(s). The larger the value of C_i^G , the more different between x_i and the group negative ideal object d_{-i}^G, while the more similar between x_i and the group positive ideal object d_{+i}^G. Therefore, the alternative(s) with the maximum group relative-closeness coefficient should be chosen as the optimal one(s).

4 Illustrative Example

In this section, an investment problem is firstly used to demonstrate the applicability and the effectiveness of our method under hesitant fuzzy environments. Then, the investment problem is also used to demonstrate the applicability and the implementation process of the developed method under interval-valued hesitant fuzzy environments. Finally, a comparison analysis with other methods is made to show the superiority of the developed methods.

Example 4.1. Let us suppose an investment company, which wants to invest a sum of money in the best option (adapted from [33-35]). There is a panel with five possible alternatives in which to invest the money: (1) x_1 is a car industry; (2) x_2 is a food company; (3) x_3 is a computer company; (4) x_4 is an arms company; (5) x_5 is a TV company. The investment company must make a decision according to the following four attributes: (1) c_1 is the risk analysis; (2) c_2 is the growth analysis; (3) c_3 is the social–political impact analysis; (4) c_4 is the environmental impact analysis. Suppose that five possible candidates x_i $\left(i=1,2,3,4,5\right)$ are to be evaluated by three decision makers d_k ($k=1,2,3$) under the above four attributes c_j ($j=1,2,3,4$). The decision makers construct, respectively, three hesitant fuzzy decision matrices $A^{(k)}=\left(a_{ij}^{(k)}\right)_{5\times 4}$ ($k=1,2,3$) listed in Tables 1-3, where $a_{ij}^{(k)}$ is a HFE denoting all the possible values, given by the decision maker d_k, for the alternative x_i under the attribute c_j.

Table 1. Hesitant fuzzy decision matrix $A^{(1)}$ provided by the decision maker d_1

1	c_1	c_2	c_3	c_4
x_1	{0.5,0.4,0.3}	{0.9,0.8,0.6}	{0.4,0.3,0.2,0.1}	{0.8,0.7,0.6,0.4,0.3}
x_2	{0.8,0.7,0.6,0.5,0.3}	{0.9,0.7,0.5,0.4}	{0.3,0.2}	{0.6,0.5,0.4,0.3}
x_3	{0.7,0.6}	{0.8,0.6,0.5}	{0.7,0.5,0.3}	{0.4,0.3}
x_4	{0.7,0.5}	{0.4,0.3}	{0.9,0.8,0.7,0.6}	{0.5,0.4,0.3}
x_5	{0.9,0.7}	{0.5,0.3}	{0.5,0.4,0.3}	{0.8,0.7,0.5}

Table 2. Hesitant fuzzy decision matrix $A^{(2)}$ provided by the decision maker d_2.

2	c_1	c_2	c_3	c_4
x_1	{0.9,0.8,0.7}	{0.4,0.3,0.2}	{0.8,0.6}	{0.7,0.6,0.5}
x_2	{0.7,0.6,0.5,0.4,0.3}	{0.8,0.7,0.6,0.5}	{0.5,0.4,0.3}	{0.8,0.7,0.6,0.4,0.3}
x_3	{0.3,0.1}	{0.5,0.3,0.2,0.1}	{0.8,0.6,0.5}	{0.9,0.8,0.7}
x_4	{0.9,0.8,0.7}	{0.7,0.6}	{0.6,0.5,0.3}	{0.8,0.6}
x_5	{0.7,0.6}	{0.8,0.7,0.4,0.3}	{0.9,0.7,0.6,0.3,0.2}	{0.5,0.4}

Table 3. Hesitant fuzzy decision matrix $A^{(3)}$ provided by the decision maker d_3.

3	c_1	c_2	c_3	c_4
x_1	{0.7,0.6,0.5,0.4,0.3}	{0.4,0.3,0.1}	{0.6,0.5,0.4}	{0.8,0.7,0.6,0.4}
x_2	{0.6,0.5,0.3}	{0.4,0.3,0.2}	{0.9,0.7}	{0.7,0.5}
x_3	{0.8,0.6,0.5}	{0.2,0.1}	{0.6,0.4,0.3,0.2,0.1}	{0.9,0.7,0.6,0.5}
x_4	{0.9,0.6}	{0.8,0.6,0.5,0.3,0.1}	{0.7,0.5,0.3}	{0.8,0.7,0.6}
x_5	{0.8,0.7,0.6}	{0.6,0.5,0.4}	{0.7,0.6,0.5}	{0.9,0.7,0.5}

In what follows, we utilize the developed method to find the best alternative(s). We now discuss two different cases.

Case 1: Assume that the information about the attribute weights is completely unknown; in this case, we use the following steps to get the most desirable alternative(s).

Step 1. Considering that all the attributes c_j ($j=1,2,3,4$) are the benefit type attributes, the hesitant fuzzy decision matrices $A^{(k)}=\left(a_{ij}^{(k)}\right)_{5\times4}$ ($k=1,2,3$) do not need normalization. Suppose that all the decision makers (DMs) ($k=1,2,3$) are pessimistic, then we utilize Definition 2.5 to transform the hesitant fuzzy decision matrices $A^{(k)}=\left(a_{ij}^{(k)}\right)_{5\times4}$ ($k=1,2,3$) into the corresponding hesitant fuzzy decision matrices $H^{(k)}=\left(h_{ij}^{(k)}\right)_{5\times4}$ ($k=1,2,3$) (see Tables 4-6), such that $l_{h_{ij}^{(k)}}=5$ for all $i=1,2,3,4,5$, $j=1,2,3,4$, and $k=1,2,3$.

Table 4. Hesitant fuzzy decision matrix $H^{(1)}$ provided by the decision maker d_1.

4	c_1	c_2	c_3	c_4
x_1	{0.5,0.4,0.3,0.3,0.3}	{0.9,0.8,0.6,0.6,0.6}	{0.4,0.3,0.2,0.1,0.1}	{0.8,0.7,0.6,0.4,0.3}
x_2	{0.8,0.7,0.6,0.5,0.3}	{0.9,0.7,0.5,0.4,0.4}	{0.3,0.2,0.2,0.2,0.2}	{0.6,0.5,0.4,0.3,0.3}
x_3	{0.7,0.6,0.6,0.6,0.6}	{0.8,0.6,0.5,0.5,0.5}	{0.7,0.5,0.3,0.3,0.3}	{0.4,0.3,0.3,0.3,0.3}
x_4	{0.7,0.5,0.5,0.5,0.5}	{0.4,0.3,0.3,0.3,0.3}	{0.9,0.8,0.7,0.6,0.6}	{0.5,0.4,0.3,0.3,0.3}
x_5	{0.9,0.7,0.7,0.7,0.7}	{0.5,0.3,0.3,0.3,0.3}	{0.5,0.4,0.3,0.3,0.3}	{0.8,0.7,0.5,0.5,0.5}

Table 5. Hesitant fuzzy decision matrix $H^{(2)}$ provided by the decision maker d_2.

5	c_1	c_2	c_3	c_4
x_1	{0.9,0.8,0.7,0.7,0.7}	{0.4,0.3,0.2,0.2,0.2}	{0.8,0.6,0.6,0.6,0.6}	{0.7,0.6,0.5,0.5,0.5}
x_2	{0.7,0.6,0.5,0.4,0.3}	{0.8,0.7,0.6,0.5,0.5}	{0.5,0.4,0.3,0.3,0.3}	{0.8,0.7,0.6,0.4,0.3}
x_3	{0.3,0.1,0.1,0.1,0.1}	{0.5,0.3,0.2,0.1,0.1}	{0.8,0.6,0.5,0.5,0.5}	{0.9,0.8,0.7,0.7,0.7}
x_4	{0.9,0.8,0.7,0.7,0.7}	{0.7,0.6,0.6,0.6,0.6}	{0.6,0.5,0.3,0.3,0.3}	{0.8,0.6,0.6,0.6,0.6}
x_5	{0.7,0.6,0.6,0.6,0.6}	{0.8,0.7,0.4,0.3,0.3}	{0.9,0.7,0.6,0.3,0.2}	{0.5,0.4,0.4,0.4,0.4}

Table 6. Hesitant fuzzy decision matrix $H^{(3)}$ provided by the decision maker d_3.

6	c_1	c_2	c_3	c_4
x_1	{0.7,0.6,0.5,0.4,0.3}	{0.4,0.3,0.1,0.1,0.1}	{0.6,0.5,0.4,0.4,0.4}	{0.8,0.7,0.6,0.4,0.4}
x_2	{0.6,0.5,0.3,0.3,0.3}	{0.4,0.3,0.2,0.2,0.2}	{0.9,0.7,0.7,0.7,0.7}	{0.7,0.5,0.5,0.5,0.5}
x_3	{0.8,0.6,0.5,0.5,0.5}	{0.2,0.1,0.1,0.1,0.1}	{0.6,0.4,0.3,0.2,0.1}	{0.9,0.7,0.6,0.5,0.5}
x_4	{0.9,0.6,0.6,0.6,0.6}	{0.8,0.6,0.5,0.3,0.1}	{0.7,0.5,0.3,0.3,0.3}	{0.8,0.7,0.6,0.6,0.6}
x_5	{0.8,0.7,0.6,0.6,0.6}	{0.6,0.5,0.4,0.4,0.4}	{0.7,0.6,0.5,0.5,0.5}	{0.9,0.7,0.5,0.5,0.5}

Step 2: Utilize Eq. (11) to get the weights of the decision makers:

$$\omega=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$$

Step 3. Considering that the information about the attribute weights is completely unknown, we utilize Eq. (24) to get the optimal weight vector of attributes:

$$w=\left(0.2694,0.2850,0.2694,0.1762\right)^T$$

Step 4. Utilize Eqs. (25) and (26) to determine the HFPIS $h_+^{(k)}$ ($k=1,2,3$) and the HFNIS $h_-^{(k)}$

($k=1,2,3$) for each decision maker d_k ($k=1,2,3$), respectively:

$$h_+^{(1)}=\{\{0.9,0.7,0.7,0.7,0.7\},\{0.9,0.8,0.6,0.6,0.6\},\{0.9,0.8,0.7,0.6,0.6\},\{0.8,0.7,0.6,0.5,0.5\}\}$$
$$h_-^{(1)}=\{\{0.5,0.4,0.3,0.3,0.3\},\{0.4,0.3,0.3,0.3,0.3\},\{0.3,0.2,0.2,0.1,0.1\},\{0.4,0.3,0.3,0.3,0.3\}\}$$
$$h_+^{(2)}=\{\{0.9,0.8,0.7,0.7,0.7\},\{0.8,0.7,0.6,0.6,0.6\},\{0.9,0.7,0.6,0.6,0.6\},\{0.9,0.8,0.7,0.7,0.7\}\}$$
$$h_-^{(2)}=\{\{0.3,0.1,0.1,0.1,0.1\},\{0.4,0.3,0.2,0.1,0.1\},\{0.5,0.4,0.3,0.3,0.2\},\{0.5,0.4,0.4,0.4,0.3\}\}$$
$$h_+^{(3)}=\{\{0.9,0.7,0.6,0.6,0.6\},\{0.8,0.6,0.5,0.4,0.4\},\{0.9,0.7,0.7,0.7,0.7\},\{0.9,0.7,0.6,0.6,0.6\}\}$$
$$h_-^{(3)}=\{\{0.6,0.5,0.3,0.3,0.3\},\{0.2,0.1,0.1,0.1,0.1\},\{0.6,0.4,0.3,0.2,0.1\},\{0.7,0.5,0.5,0.4,0.4\}\}$$

Step 5: Utilize Eqs. (27) and (28) to calculate the separation measures $d_{+i}^{(k)}$ and $d_{-i}^{(k)}$ of each alternative x_i of the decision maker d_k:

$d_{+1}^{(1)}=0.2477$, $d_{-1}^{(1)}=0.1613$, $d_{+2}^{(1)}=0.2473$, $d_{-2}^{(1)}=0.1618$, $d_{+3}^{(1)}=0.2002$, $d_{-3}^{(1)}=0.2088$, $d_{+4}^{(1)}=0.2080$, $d_{-4}^{(1)}=0.2010$, $d_{+5}^{(1)}=0.2031$, $d_{-5}^{(1)}=0.2059$,
$d_{+1}^{(2)}=0.1600$, $d_{-1}^{(2)}=0.2875$, $d_{+2}^{(2)}=0.2029$, $d_{-2}^{(2)}=0.2446$, $d_{+3}^{(2)}=0.3137$, $d_{-3}^{(2)}=0.1338$, $d_{+4}^{(2)}=0.1080$, $d_{-4}^{(2)}=0.3395$, $d_{+5}^{(2)}=0.1809$, $d_{-5}^{(2)}=0.2665$,
$d_{+1}^{(3)}=0.2384$, $d_{-1}^{(3)}=0.1016$, $d_{+2}^{(3)}=0.1799$, $d_{-2}^{(3)}=0.1601$, $d_{+3}^{(3)}=0.2668$, $d_{-3}^{(3)}=0.0732$, $d_{+4}^{(3)}=0.1179$, $d_{-4}^{(3)}=0.2221$, $d_{+5}^{(3)}=0.0873$, $d_{-5}^{(3)}=0.2527$.

Step 6: Utilize Eq. (29) to calculate the relative closeness coefficient $C_i^{(k)}$ of each alternative x_i to the HFPIS $h_+^{(k)}$ of the decision maker d_k as

$$C_1^{(1)}=0.3945,\ C_2^{(1)}=0.3955,\ C_3^{(1)}=0.5105,\ C_4^{(1)}=0.4915,\ C_5^{(1)}=0.5034,$$
$$C_1^{(2)}=0.6424,\ C_2^{(2)}=0.5465,\ C_3^{(2)}=0.2990,\ C_4^{(2)}=0.7587,\ C_5^{(2)}=0.5956,$$
$$C_1^{(3)}=0.2987,\ C_2^{(3)}=0.4709,\ C_3^{(3)}=0.2152,\ C_4^{(3)}=0.6532,\ C_5^{(3)}=0.7434.$$

Then, we construct the relative-closeness coefficient matrix as below:

$$C=\begin{pmatrix}0.3945 & 0.6424 & 0.2987\\ 0.3955 & 0.5465 & 0.4709\\ 0.5105 & 0.2990 & 0.2152\\ 0.4915 & 0.7587 & 0.6532\\ 0.5034 & 0.5956 & 0.7434\end{pmatrix}_{5\times 3}$$

Step 7. Utilize Eqs. (31) and (32) to identify the group positive ideal solution (GPIS) and group negative ideal solution (GNIS), respectively, as follows:

$$h_+^G=\{0.5105,0.7587,0.7434\}$$

$$h_-^G = \{0.3945, 0.2990, 0.2152\}$$

Step 8. Utilize Eqs. (33) and (34) to calculate the separation measures d_{+i}^G and d_{-i}^G of each alternative x_i from the group positive ideal solution h_+^G and the group negative ideal solution h_-^G, respectively, as follows:

$d_{+1}^G = 0.2257$, $d_{-1}^G = 0.1423$, $d_{+2}^G = 0.1999$, $d_{-2}^G = 0.1681$, $d_{+3}^G = 0.3293$, $d_{-3}^G = 0.0387$, $d_{+4}^G = 0.0364$, $d_{-4}^G = 0.3316$, $d_{+5}^G = 0.0567$, $d_{-5}^G = 0.3113$

Step 9. Utilize Eq. (35) to calculate the group relative-closeness coefficient C_i^G of each alternative x_i to group positive ideal solution d_{+i}^G as:

$$C_1^G = 0.3868,\ C_2^G = 0.4568,\ C_3^G = 0.1051,\ C_4^G = 0.9011,\ C_5^G = 0.8459$$

Step 10: Rank the alternatives x_i ($i = 1,2,3,4,5$) according to the group relative-closeness coefficient C_i^G ($i = 1,2,3,4,5$). Clearly, $x_4 \succ x_5 \succ x_2 \succ x_1 \succ x_3$, and thus the best alternative is x_4.

Case 2: The information about the attribute weights is partly known and the known weight information is given as follows:

$$\Delta = \left\{ 0.15 \le w_1 \le 0.25,\ 0.2 \le w_2 \le 0.25,\ 0.3 \le w_3 \le 0.4,\ 0.35 \le w_4 \le 0.5,\ w_j \ge 0,\ j = 1,2,3,4,\ \sum_{j=1}^{4} w_j = 1 \right\}$$

Step 1'. See Step 1.
Step 2': See Step 2.
Step 3': Utilize the model (M-5) to construct the single-objective model as follows:

$$\begin{cases} \max D(w) = 4.1600 w_1 + 4.4000 w_2 + 4.1600 w_3 + 2.7200 w_4 \\ \text{s.t.} \quad w \in \Delta \end{cases}$$

By solving this model, we get the optimal weight vector of attributes $w = (0.1500, 0.2000, 0.3000, 0.3500)^T$.

Step 4'. See Step 4.
Step 5': Utilize Eqs. (27) and (28) to calculate the separation measures $d_{+i}^{(k)}$ and $d_{-i}^{(k)}$ of each alternative x_i of the decision maker d_k:

$d_{+1}^{(1)} = 0.2280$, $d_{-1}^{(1)} = 0.1720$, $d_{+2}^{(1)} = 0.2680$, $d_{-2}^{(1)} = 0.1320$, $d_{+3}^{(1)} = 0.2370$, $d_{-3}^{(1)} = 0.1630$, $d_{+4}^{(1)} = 0.1970$, $d_{-4}^{(1)} = 0.2030$, $d_{+5}^{(1)} = 0.1870$, $d_{-5}^{(1)} = 0.2130$,

$d_{+1}^{(2)} = 0.1620$, $d_{-1}^{(2)} = 0.2470$, $d_{+2}^{(2)} = 0.2130$, $d_{-2}^{(2)} = 0.1960$, $d_{+3}^{(2)} = 0.2070$, $d_{-3}^{(2)} = 0.2020$, $d_{+4}^{(2)} = 0.1340$, $d_{-4}^{(2)} = 0.2750$, $d_{+5}^{(2)} = 0.2140$, $d_{-5}^{(2)} = 0.1950$,
$d_{+1}^{(3)} = 0.2140$, $d_{-1}^{(3)} = 0.1010$, $d_{+2}^{(3)} = 0.1470$, $d_{-2}^{(3)} = 0.1680$, $d_{+3}^{(3)} = 0.2390$, $d_{-3}^{(3)} = 0.0760$, $d_{+4}^{(3)} = 0.1220$, $d_{-4}^{(3)} = 0.1930$, $d_{+5}^{(3)} = 0.0940$, $d_{-5}^{(3)} = 0.2210$.

Step 6': Utilize Eq. (29) to calculate the relative closeness coefficient $C_i^{(k)}$ of each alternative x_i to the hesitant fuzzy linguistic PIS $X_+^{(k)}$ of the decision maker d_k as

$$C_1^{(1)} = 0.4300,\ C_2^{(1)} = 0.3300,\ C_3^{(1)} = 0.4075,\ C_4^{(1)} = 0.5075,\ C_5^{(1)} = 0.5325,$$
$$C_1^{(2)} = 0.6039,\ C_2^{(2)} = 0.4792,\ C_3^{(2)} = 0.4939,\ C_4^{(2)} = 0.6724,\ C_5^{(2)} = 0.4768,$$
$$C_1^{(3)} = 0.3206,\ C_2^{(3)} = 0.5333,\ C_3^{(3)} = 0.2413,\ C_4^{(3)} = 0.6127,\ C_5^{(3)} = 0.7016.$$

Then, we construct the relative-closeness coefficient matrix as below:

$$C = \begin{pmatrix} 0.4300 & 0.6039 & 0.3206 \\ 0.3300 & 0.4792 & 0.5333 \\ 0.4075 & 0.4939 & 0.2413 \\ 0.5075 & 0.6724 & 0.6127 \\ 0.5325 & 0.4768 & 0.7016 \end{pmatrix}_{5\times 3}$$

Step 7'. Utilize Eqs. (31) and (32) to identify the group positive ideal solution (GPIS) and group negative ideal solution (GNIS), respectively as follows:

$$h_+^G = \{0.5325, 0.6724, 0.7016\}$$
$$h_-^G = \{0.3300, 0.4768, 0.2413\}$$

Step 8'. Utilize Eqs. (33) and (34) to calculate the separation measures d_{+i}^G and d_{-i}^G of each alternative x_i from the GPIS h_+^G and the GNIS h_-^G , respectively, as follows:
$d_{+1}^G = 0.1840$, $d_{-1}^G = 0.1022$, $d_{+2}^G = 0.1880$, $d_{-2}^G = 0.0982$, $d_{+3}^G = 0.2546$, $d_{-3}^G = 0.0315$, $d_{+4}^G = 0.0380$, $d_{-4}^G = 0.2482$, $d_{+5}^G = 0.0652$, $d_{-5}^G = 0.2209$

Step 9'. Utilize Eq. (35) to calculate the group relative-closeness coefficient C_i^G of each alternative x_i to group positive ideal solution d_{+i}^G as:

$$C_1^G = 0.3571,\ C_2^G = 0.3431,\ C_3^G = 0.1102,\ C_4^G = 0.8673,\ C_5^G = 0.7721$$

Step 10': Rank the alternatives x_i ($i = 1,2,3,4,5$) according to the group relative-closeness coefficient C_i^G ($i = 1,2,3,4,5$). Clearly, $x_4 \succ x_5 \succ x_1 \succ x_2 \succ x_3$, and thus the best alternative is x_4 .

5 Comparison Analysis with the Other Hesitant Fuzzy Multiple Attribute Decision Making (Madm) Methods

In this section, we will perform a comparison analysis between our new method and the other existing hesitant fuzzy multiple attribute decision making methods, and then highlight the advantages of the new method over the other existing methods.

5.1 Comparison with the Hesitant Fuzzy MADM Methods Based on TOPSIS

Zhang and Wei [36] extended the TOPSIS method to develop a methodology for solving MADM problems with hesitant fuzzy information. Recently, Xu and Zhang [26] developed a method based on TOPSIS and the maximizing deviation method for solving MADM problems with hesitant fuzzy information, in which the attribute values provided by the decision makers are expressed in hesitant fuzzy elements and the information about attribute weights is incomplete. Moreover, they extended the developed method to interval-valued hesitant fuzzy situations. Compared with Zhang and Wei's method and Xu and Zhang's method, the newly developed method has the following advantages: Zhang and Wei's method and Xu and Zhang's method focus on the MADM problems. However, in real-life, due to the increasing complexity of socio-economic environment, it is less and less possible for a single decision maker to consider all relevant aspects of the problem. Therefore, many organizations employ groups to make decision, which is called as group decision making (GDM). Our method gives a TOPSIS based procedure to solve a MAGDM problem under hesitant fuzzy environments. First, in our method, a quadratic programming model is established to determine the weights of decision makers, which is not be considered in Zhang and Wei's method [36] and Xu and Zhang's method [26]. Second, Zhang and Wei's method [36] doesn't consider the weights of attributes. Though Xu and Zhang [26] established an optimization model to determine the attribute weights, this model determined the attribute weights from only an individual hesitant fuzzy decision matrix, and it cannot determine the importance weights of attributes under group decision making environments. Our method can derive the optimal weights of attributes from all individual hesitant fuzzy decision matrices. Finally, the TOPSIS methods proposed by Zhang and Wei [36] and Xu and Zhang [26] only included a stage; while the extended TOPSIS proposed by our method includes two stages: The first stage is called the hesitant fuzzy TOPSIS (HFTOPSIS), which can be used to calculate the individual relative closeness coefficient of each alternative to the individual hesitant fuzzy PIS. The second stage is the standard TOPSIS, which is used to calculate the group relative-closeness coefficient of each alternative to group PIS and select the optimal one with the maximum group relative-closeness coefficient.

5.2 Comparison with the Hesitant Fuzzy MADM Methods Based on Hesitant Fuzzy Aggregation Operators

Recently, some hesitant fuzzy aggregation operators have been developed for aggregating hesitant fuzzy information [19-24], such as the HFWA, HFWG, GHFWA, GHFWG, HFOWA, HFOWG, GHFOWA, GHFOWG, HFHA, HFHG, GHFHA, GHFHG, HFPWA, HFPWG, HFPA, HFPG, GHFPA, GHFPG, WGHFPA, WGHFPG, HFPOWA, HFPOWG, GHFPOWA, GHFPOWG, HFPWA, and HFPWG operators. Furthermore, based on these operators, some hesitant fuzzy MADM methods [19-24] have also been developed for solving the MADM or MAGDM problems with hesitant fuzzy information. However, it is noticed that these existing operators and methods have some inherent drawbacks, which are shown as follows:

(1) The existing operators and methods perform an aggregation on the input hesitant fuzzy arguments. Accordingly, the dimension of the derived HFE may increase as such an aggregation is done, which may increase the computational complexity and therefore lead to the loss of information. In contrast, our method does not need to perform such an aggregation but directly deals with the input hesitant fuzzy arguments, thereby does not increase the dimension of the derived HFE and retains the original decision information as much as possible.

(2) Our method utilizes the maximizing group consensus method and the maximizing deviation method to determine the weight values of decision makers and attributes, respectively, which is more objective and reasonable; while the existing methods [19-24] ask the DMs to provide the weights of decision makers and attributes in advance, which is subjective and sometime cannot yield the persuasive results.

In order to clearly demonstrate the comparison results, we use the hesitant fuzzy weighted averaging (HFWA) operator-based MAGDM method [20] to revisit Example 4.1, which includes the following steps:

Step 1: Utilize the HFWA operator [20]:

$$\text{HFWA}\left(a_{ij}^{(1)}, a_{ij}^{(2)}, a_{ij}^{(3)}\right) = \overset{3}{\underset{k=1}{\oplus}}\left(\omega_k a_{ij}^{(k)}\right) = \bigcup_{t_1=1,2,\cdots,l_{a_{ij}^{(1)}}, t_2=1,2,\cdots,l_{a_{ij}^{(2)}}, t_3=1,2,\cdots,l_{a_{ij}^{(3)}}} \left\{1-\prod_{k=1}^{3}\left(1-\left(a_{ij}^{(k)}\right)^{t_k}\right)^{\omega_k}\right\}$$

$$i = 1,2,3,4,5,\ \ j = 1,2,3,4.$$

to aggregate all the individual hesitant fuzzy decision matrix $A^{(k)} = \left(a_{ij}^{(k)}\right)_{5\times4}$ ($k = 1,2,3$) into the collective hesitant fuzzy decision matrix $A = \left(a_{ij}\right)_{5\times4}$, which is not be listed here because of space limitations. In order to be consistent with Example 5.1, the same weights for decision makers obtained, i.e., $\omega_1 = \dfrac{1}{3}$, $\omega_2 = \dfrac{1}{3}$, and $\omega_3 = \dfrac{1}{3}$ are adopted here. Let $L = \left(l_{a_{ij}}\right)_{5\times4}$, where $l_{a_{ij}}$ is the dimension of the collective hesitant fuzzy element a_{ij}.

$$L = \left(l_{a_{ij}}\right)_{5\times4} = \begin{pmatrix} 45 & 27 & 24 & 60 \\ 75 & 48 & 12 & 40 \\ 12 & 24 & 45 & 24 \\ 12 & 20 & 36 & 18 \\ 12 & 24 & 45 & 18 \end{pmatrix}$$

Step 2. Utilize the HFWA operator [23]:

$$\text{HFWA}\left(a_{i1}, a_{i2}, a_{i3}, a_{i4}\right) = \overset{4}{\underset{j=1}{\oplus}}\left(w_j a_{ij}\right) = \bigcup_{t_1=1,2,\cdots,l_{a_{i1}}, t_2=1,2,\cdots,l_{a_{i2}}, t_3=1,2,\cdots,l_{a_{i3}}, t_3=1,2,\cdots,l_{a_{i4}}} \left\{1-\prod_{j=1}^{4}\left(1-\left(a_{ij}\right)^{t_j}\right)^{w_j}\right\}$$

$$(i = 1,2,3,4,5)$$

to aggregate all the preference values a_{ij} $\left(j=1,2,3,4\right)$ in the *i*th line of A, and then derive the collective overall preference value a_i $\left(i=1,2,3,4,5\right)$ of the alternative x_i ($i=1,2,3,4,5$). In order to be consistent with Example 4.1, the same weights for attributes obtained, i.e., $w_1=0.2694$, $w_2=0.2850$, $w_3=0.2694$, and $w_4=0.1762$ are adopted here. We will not list the collective overall preference values here because of space limitations. The dimensions of the collective overall preference value a_i $\left(i=1,2,3,4,5\right)$ are shown bellows:

$$l_{a_1}=1749600, \quad l_{a_2}=1728000, \quad l_{a_3}=311040, \quad l_{a_4}=155520, \quad l_{a_5}=233280$$

Step 3. According to Definition 2.2, we calculate the score values $s\left(a_i\right)$ ($i=1,2,3,4,5$) of a_i ($i=1,2,3,4,5$):

$$s\left(a_1\right)=0.5747, \quad s\left(a_2\right)=0.5651, \quad s\left(a_3\right)=0.5328, \quad s\left(a_4\right)=0.6422, \quad s\left(a_5\right)=0.6206$$

Step 4. Get the priority of the alternatives x_i ($i=1,2,3,4,5$) by ranking $s\left(a_i\right)$ ($i=1,2,3,4,5$) as follows: $x_4 \succ x_5 \succ x_1 \succ x_2 \succ x_3$. Thus, the best alternative is x_4.

It is easy to see that the optimal alternative obtained by the Xia and Xu' method is the same as our method, which shows the effectiveness, preciseness, and reasonableness of our method. However, it is noticed that the ranking order of the alternatives obtained by our method is $x_4 \succ x_5 \succ x_2 \succ x_1 \succ x_3$, which is different from the ranking order obtained by the Xia and Xu' method. Concretely, the ranking order between x_1 and x_2 obtained by the two methods are just converse, i.e., $x_2 \succ x_1$ for our method while $x_1 \succ x_2$ for the Xia and Xu' method. The main reason is that the Xia and Xu' method performs an aggregation operation on the input hesitant fuzzy arguments, while our method does need to perform such an operation on the input hesitant fuzzy arguments. It is noted that the dimension l_{a_i} of the collective overall preference value a_i is very larger, which increases the computational complexity. In contrast, our method has a less computational complexity. By using the MATLAB mathematics software under the same conditions, the time (12 hours) that is used to obtain the optimal alternative with the Xia and Xu' method is far more than the one (1 second) that is used to obtain the optimal alternative with our method. Therefore, our method not only is appropriate for handling the situations in which the weight information of the attributes and decision makers is unknown or partly known, but also can reduce the computational complexity and the information loss, which always happens in the process of information aggregation. Thus, compared with the other hesitant fuzzy MADM methods, our method has its great superiority in dealing with the ambiguity and hesitancy inherent in MAGDM problems with hesitant fuzzy information.

6 Conclusion

In this paper, we have proposed a novel method for hesitant fuzzy MAGDM problems with incomplete weight information, which involves three parts:

(1) First, inspired by the idea that a set of group members should have a maximum degree of

agreement solution, we have first used the maximizing the group consensus method to establish a quadratic programming model for determining the optimal weights of decision makers under hesitant fuzzy situations. This part ensures the rationality of the individual hesitant fuzzy decision information.

(2) Then, based on the idea that a larger weight should be assigned to the attribute with a larger deviation value among alternatives, we have further presented a maximizing deviation method to determine the optimal attribute weights under hesitant fuzzy environments. This part eliminates the influence of subjectivity of attribute weights provided by the decision makers in advance.

(3) Furthermore, we have proposed an extended TOPSIS method for solving MAGDM problems with hesitant fuzzy information, which includes two stages: the HFTOPSIS and the standard TOPSIS. The former is used to calculate the relative closeness coefficient of each alternative to the HFPIS; while the latter is used to calculate the group relative-closeness coefficient of each alternative to GPIS, based on which we rank the considered alternatives and then select the optimal one with the maximum group relative-closeness coefficient. The extended TOPSIS method not only solves a MAGDM problem with hesitant fuzzy information but also can avoid the loss of hesitant fuzzy information in the process of information aggregation.

Finally, an investment example has been used to illustrate the effectiveness and practicality of the developed methods. A comparison analysis has also been conducted to illustrate the advantages of the developed methods over the other hesitant fuzzy MADM methods.

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Authors' Contributions

'Zhiming Zhang' designed the study, performed the statistical analysis, wrote the protocol, and wrote the overall draft of the manuscript. The author read and approved the final manuscript.

Competing Interests

Author has declared that no competing interests exist.

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