

**British Journal of Mathematics & Computer Science 4(13): 1849-1856, 2014**



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# **Periodic Travelling Wave Solutions of ZK (2, 4, -2) Equation**

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*Original Research Article*

*Received: 20 February 2014 Accepted: 13 April 2014 Published: 08 May 2014*

### **Abstract**

In this paper, the qualitative analysis methods of dynamical systems are used to investigate the periodic travelling wave solutions of  $ZK$  (2, 4,-2) equation. The phase portrait bifurcation of the travelling wave system corresponding to the equation is given. The explicit expressions of the periodic travelling wave solutions are obtained by using the portraits. The graph of the solutions are given with the numerical simulation.

Keywords: Periodic travelling wave solutions, ZK (2, 4, -2) equation, bifurcation method. 2010 Mathematics Subject Classification: 53C25; 83C05; 57N16

## **1 Introduction**

The study of travelling wave solutions in particular, solitons, of Partial Deffierential Equations (what called PDEs), for various nonlinear evolution equations in mathematical physics plays an important role in soliton theory. To obtain the travelling wave solutions for PDEs, a lot of systematic methods have been developed for soliton equations, such as the inverse scattering method, the B*a*cklund and the Darboux transformations, the tanh-function method, the homogeneous balance method, the extended tanh-function method and others [1-7,18].

The ZK equation governs the behavior of weakly nonlinear ion-acoustic waves in aplasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [8]. The ZK equation, which is a more isotropic two-dimensional, was first derived for describing weakly nonlinear ion-acoustic waves in a strongly magnetized lossless plasma in two dimension [9].

Recently, the following ZK equation

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$$
u_t + auu_x + b(u_{xx} + u_{yy})_x = 0,
$$
\n(1.1)

was investigated by BK. Shivamoggi [10], A. M. Wazwaz [11] and VE. Zakharov etc [9] with various distinct approaches.

Eritish Journal of Mathematics & Computer Science 4(13), 1849-1856, 2014<br>  $u_t + auu_x + b(u_{xx} + u_{yy})_x = 0,$  (1.1)<br>
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ggi [10], A. M. Wazwaz [11] and VE. Zakharov etc [9] with<br>
cosine method and the tanh method, A. M. Wazwaz [12]<br>
tion,<br>  $(u^n)_$ *British Journal of Mathematics & Computer Science 4(13), 1849-1856, 2014*<br> *t*<sub>*t*</sub> + *atti*<sub>*x*</sub> + *b*(*u*<sub>*xx*</sub> + *u*<sub>*y*</sub> )<sub>*x*</sub> = 0, (1.1)<br>
ivamoggi [10], A. M. Wazwaz [11] and VE. Zakharov etc [9] with<br>
e sine-cosin

$$
u_t + a(u^n)_x + b[u^{-n}(u^{2n})_{xx} + k(u^n)_{yy}]_x = 0,
$$
\n(1.2)

and the  $ZK$  (n, 2n, -n) equation,

$$
u_t + a(u^n)_x + b[u^{2n}(u^{-n})_{xx} + k(u^n)_{yy}]_x = 0,
$$
\n(1.3)

*Britsb, Journal of Mathematics & Compare Science 4(13), 1849-1856, 2014*  
\n
$$
u_t + auu_x + b(u_{xx} + u_{yy})_x = 0,
$$
 (1.1)  
\nwas investigated by RK. Shivamoggi [10], A. M. Wazwaz [11] and VF. Zakharov etc [9] with  
\nvarious distinct approaches.  
\nMore recently, by using the sine-cosine method and the tanh method, A. M. Wazwaz [12]  
\ninvestigated the ZK (n, -n, 2n) equation,  
\n $u_t + a(u'')_x + b[u^{2n}(u^{2n})_{xx} + k(u'')_{yy}]_x = 0,$  (1.2)  
\nand the ZK (n, 2n, -n) equation,  
\n $u_t + a(u'')_x + b[u^{2n}(u^{2n})_{xx} + k(u'')_{yy}]_x = 0,$  (1.3)  
\nWhere  $a, b, k$  are three non-zero real numbers, and obtained a family of solutions:  
\n $u_t = \begin{cases} \frac{2ac(b+k)}{a(b(a-n)-1+k(n+1))} \sin^2(\sqrt{\frac{n-1}{2n}}(x+y-ct))^{\frac{1}{n-1}} |A_5|^2 \le \pi, \\ 0, otherwise, \end{cases}$   
\n $u_2 = \begin{cases} \frac{2ac(b+k)}{a(b(n-1)+k(n+1))} \cos^2(\sqrt{\frac{n-1}{2n}}(x+y-ct))^{\frac{1}{n-1}} |A_5|^2 \le \pi, \\ 0, otherwise, \end{cases}$   
\nother exact explicit solutions were listed in [13]. However, the bifurcation behavior of the  
\ntravelling wave equations for corresponding traveling wave equations have equations have functions which which in its  
\nparameters space. It is very important to understand the dynamic behavior of solutions for the  
\ntravelling wave equations for corresponding variables in this paper, we shall continue to study and obtain the periodic traveling wave  
\nsolutions of the, (1.3) for  $n = 2$  (similar) equal to this method of dynamical systems [13,14] to investigate the following equation, which employs bifurcation  
\nmethod of dynamical systems [13,14] to investigate the following equation, which employs bifurcation  
\nmethod of dynamical systems [13,14] to investigate the following equation, which employs bifurcation  
\n $u_t + a(u^2)_x + b[u^4(u^{-2})_{xy} + k(u^2)_y]_x = 0,$  (1.4)  
\nTaking the transformation  $u(x, y, t) = \phi(x + y$ 

other exact explicit solutions were listed in [13]. However, the bifurcation behavior of the travelling wave solutions for corresponding travelling wave equations haven't studied in its parameter space. It is very important to understand the dynamical behavior of solutions for the travelling wave equations, the related investigation of  $(1.2)$  and  $(1.3)$  hasn't been mentioned in the literatures. In this paper, we shall continue to study and obtain the periodic travelling wave solutions of Eq. (1.3) for  $n = 2$  (simply called *Z K*(2,4,2) equation), which employs bifurcation method of dynamical systems [13,14] to investigate the following equation:

$$
u_t + a(u^2)_x + b[u^4(u^{-2})_{xx} + k(u^2)_{yy}]_x = 0,
$$
\n(1.4)

(1.4) becomes

$$
-c\phi' + a(u^2)' + b[\phi^4(\phi^{-2})'' + k(u^2)'''] = 0,
$$
\n(1.5)

where "" is the derivative with respect to  $\xi$ . Taking the integration once on both sides leads to

$$
-c\phi + a(u^{2})' + b[\phi^{4}(\phi^{-2})'' + k(u^{2})''] = g,
$$
\n(1.6)

where  $g$  is the integration constant. Clearly,  $(1.6)$  is equivalent to the following two-dimensional system

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$$
-c\phi + a(u^2)' + b[\phi^4(\phi^{-2})'' + k(u^2)'''] = g,\tag{1.6}
$$
\nconstant. Clearly, (1.6) is equivalent to the following two-dimensional\n
$$
\frac{d\phi}{d\xi} = y,\quad \frac{dy}{d\xi} = \frac{g + c\phi - a\phi^2 - py^2}{q\phi},\tag{1.7}
$$
\n
$$
k - 2b,\text{ which has the first integral}
$$
\n
$$
y^2 = \phi^{\frac{2p}{q}}(\frac{g}{p}\phi^{\frac{2p}{q}} + \frac{2c}{2p+d\phi}\phi^{\frac{2p}{q}+1} - \frac{a}{p+d\phi}\phi^{\frac{2p}{q}+2} + h)\tag{1.8}
$$

*British Journal of Mathematics & Computer Science 4(13), 1849-1856, 2014*\n
$$
-c\phi + a(u^2)' + b[\phi^4(\phi^{-2})'' + k(u^2)''] = g, \qquad (1.6)
$$
\nwhere *g* is the integration constant. Clearly, (1.6) is equivalent to the following two-dimensional system\n
$$
\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{g + c\phi - a\phi^2 - py^2}{q\phi}, \qquad (1.7)
$$
\nwhere  $p = 6b + 2k, q = 2k - 2b$ , which has the first integral\n
$$
y^2 = \phi^{-\frac{2p}{q}}(\frac{g}{\rho}\phi^q + \frac{2c}{2p+q}\phi^{\frac{2p}{q-1}} - \frac{a}{p+q}\phi^{\frac{2p}{q-2}} + h) \qquad (1.8)
$$
\nOr\n
$$
H(\phi, y) = \phi^{-\frac{2p}{q}} y^2 - \phi^{-\frac{2p}{q}}(\frac{g}{p} + \frac{2c}{2p+q}\phi - \frac{a}{p+q}\phi^2) = h. \qquad (1.9)
$$
\nObviously system (1.7) is a five-parameter planar dynamical system depending on the parameter group (*e*, *c*, *a*, *n*, *a*). Since the phase orbits defined by the vector fields of Eq. (1.7) determine all

Or

$$
H(\phi, y) = \phi^{\frac{2p}{q}} y^2 - \phi^{\frac{2p}{q}} \left( \frac{g}{p} + \frac{2c}{2p+q} \phi - \frac{a}{p+q} \phi^2 \right) = h \,. \tag{1.9}
$$

*ritish Journal of Mathematics & Computer Science 4(13), 1849-1856, 2014*<br>  $\phi + a(u^2)' + b[\phi^4(\phi^{-2})'' + k(u^2)'''] = g,$  (1.6)<br>
stant. Clearly, (1.6) is equivalent to the following two-dimensional<br>  $\frac{\phi}{\overline{z}} = y$ ,  $\frac{dy}{d\overline{\zeta}} = \frac$ *British Journal of Mathematics & Computer Science 4(13), 1849-1856, 2014*<br>  $-\epsilon \phi + a(u^2)' + b[\phi^4 (\phi^{-2})'' + k(u^2)''] = g,$  (1.6)<br>
onstant. Clearly, (1.6) is equivalent to the following two-dimensional<br>  $\frac{d\phi}{d\xi} = y$ ,  $\frac{dy}{d\xi} = \frac$ *p*  $\left(\frac{a}{p}\right)^2 + k(u^2)^n = g$ , (1.6)<br> *p*  $\left(\frac{a}{p}\right)^2 = g$ , (1.6)<br> *p*  $\left(\frac{a}{p}\right)^2 = g$ , (1.6)<br> *p*  $\left(\frac{a}{p}\right)^2 - \frac{a}{p^2}$ , (1.7)<br> *p*  $\left(\frac{a}{p}\right)^2 - \frac{a}{p+q}$   $\left(\frac{a}{p}\right)^2 + h$  (1.8)<br> *p*  $\left(\frac{a}{p}\right)^2 + \frac{a}{p+q}$   $\left(\frac{a}{$ *itish Journal of Mathematics & Computer Science 4(13), 1849-1856, 2014*<br>  $\phi + a(u^2)' + b[\phi^4(\phi^{-2})' + k(u^2)'] = g,$  (1.6)<br>
stant. Clearly, (1.6) is equivalent to the following two-dimensional<br>  $= y, \frac{dy}{d\zeta} = \frac{g + c\phi - a\phi^2 - py^2}{g\phi$ puter Science 4(13), 1849-1856, 2014<br>
<sup>2</sup>)<sup>*r*</sup>] = *g*, (1.6)<br>
to the following two-dimensional<br>
<u>ppy<sup>2</sup></u></sup>, (1.7)<br>  $-\frac{a}{p+q} \phi^{\frac{2p}{q}+2} + h$ ) (1.8)<br>  $\frac{c}{p+q} \phi - \frac{a}{p+q} \phi^2$ ) = *h*. (1.9)<br>
system depending on the par Obviously system (1.7) is a five-parameter planar dynamical system depending on the parameter  $-c\phi + a(u^2)' + b[\phi^4(\phi^{-2})'' + k(u^2)^n] = g$ , (1.6)<br>
where g is the integration constant. Clearly, (1.6) is equivalent to the following two-dimensional<br>
system<br>  $\frac{d\phi}{d\xi} = y$ ,  $\frac{dy}{d\xi} = \frac{g + c\phi - a\phi^2 - py^2}{q\phi}$ , (1.7)<br>
where  $p =$ traveling wave solutions of system (1.3), we should investigate the bifurcations of phase portraits where *g* is the integration constant. Clearly, (1.6) is equivalent to the following two-dimensional<br>system<br>system<br> $\frac{d\phi}{d\xi} = y$ ,  $\frac{dy}{d\xi} = \frac{g + c\phi - a\phi^2 - p y^2}{q\phi}$ , (1.7)<br>where  $p = 6b + 2k$ ,  $q = 2k - 2b$ , which has the  $y$ ) =  $\phi^{\frac{2p}{q}} y^2 - \phi^{\frac{2p}{q}} (\frac{g}{p} + \frac{2c}{2p+q} \phi - \frac{a}{p+q} \phi^2) = h.$  (1.9)<br>
arameter planar dynamical system depending on the parameter<br>
asse orbits defined by the vector fields of Eq. (1.7) determine all<br>
(1.3), we  $-\phi \frac{2p}{q} \frac{g}{r} + \frac{2c}{2p+q} \phi - \frac{a}{p+q} \phi^2$  = h. (1.9)<br>
aar dynamical system depending on the parameter<br>
fined by the vector fields of Eq. (1.7) determine all<br>
ould investigate the bifurcations of phase portraits<br>
par  $+\frac{2}{2p+q}\phi^q - \frac{d}{p+q}\phi^q + h$  (1.8)<br>  $\frac{2p}{p+q}\phi^q - \frac{a}{p+q}\phi^q + h$  (1.8)<br>  $\frac{2p}{p+q}\phi^q - \frac{a}{p+q}\phi^q = \frac{a}{p+q}\phi^q$ <br>  $\frac{2c}{p+q}\phi - \frac{a}{p+q}\phi^q = h$ . (1.9)<br>
alanar dynamical system depending on the parameter<br>
defined by th  $H(\phi, y) = \phi^{\frac{2z}{\theta}} y^2 - \phi^{\frac{2z}{\theta}} (\frac{g}{\rho} + \frac{2c}{2p+q}\phi - \frac{a}{p+q}\phi^2) = h.$  (1.9)<br>Obviously system (1.7) is a five-parameter planar dynamical system depending on the parameter<br>proup (*g*, *c*, *a*, *p*, *q*). Since the ph  $H(\phi, y) = \phi^T y^2 - \phi^T y^2 = \phi^T (\frac{\beta}{\beta} + \frac{2c}{2\beta + q} \phi - \frac{d}{q} \phi^2) = h.$  (1.9)<br>
rotonsly system (1.7) is a five-parameter planar dynamical system depending on the parameter<br>  $p(\{g, c, a, p, q\})$ . Since the phase orbits defined by

The rest of this paper is organized as follows: in Section 2, we discuss the bifurcations of phase portraits of system (1.7), where explicit parametric conditions will be derived. In Section 3, we give exact explicit parametric representations for periodic solutions of Eq. (1.4) for  $n = 2$ . Section 4 contains the concluding remarks. as the parameters  $g, c, a, p$  and  $q$  are changed.<br>
lows: in Section 2, we discuss the bifurcations of phase<br>
parametric conditions will be derived. In Section 3, we<br>
tations for periodic solutions of Eq. (1.4) for  $n = 2$ .<br>

#### **2 Bifurcations and Phase Portraits of System (1.7)**

In this section, we discuss the existence of periodic solutions of  $(1.3)$  by the bifurcation method [14-17]. System (1.7) has the same phase orbits as the following system

$$
\frac{d\phi}{d\tau} = q\phi y, \frac{dy}{d\tau} = g + c\phi - a\phi^2 - py^2,
$$
\n(2.1)

*British Journal of Mathematics & Computer Science 4(13), 1849-1856, 2014*\n
$$
A_1(\phi_1, 0), A_2(\phi_2, 0) \text{ and } S_{\pm}(0, \pm Y), \text{ where}
$$
\n
$$
\phi_1 = \frac{-c + \sqrt{c^2 + 4ag}}{-2a}, \phi_2 = \frac{c + \sqrt{c^2 + 4ag}}{2a} \text{ and } Y = \sqrt{\frac{g}{p}}.
$$
\n
$$
\text{defined by (1.9), we denote that}
$$
\n
$$
0) = -\phi_i^{\frac{2p}{q}} \left(\frac{g}{p} + \frac{2c}{2p + a}\phi_i - \frac{a}{p + a}\phi_i^2\right), i = 1, 2, \ h_s = H(0, \pm Y) = 0.
$$

For the function defined by (1.9), we denote that

*British Journal of Mathematics & Computer Science 4(13), 1849-1856, 2014*\n
$$
A_1(\phi_1, 0), A_2(\phi_2, 0) \text{ and } S_{\pm}(0, \pm Y), \text{ where}
$$
\n
$$
\phi_1 = \frac{-c + \sqrt{c^2 + 4ag}}{-2a}, \phi_2 = \frac{c + \sqrt{c^2 + 4ag}}{2a} \text{ and } Y = \sqrt{\frac{g}{p}}.
$$
\nFor the function defined by (1.9), we denote that\n
$$
h_i = H(\phi_1, 0) = -\phi_i^{\frac{2p}{q}} \left( \frac{g}{p} + \frac{2c}{2p+q} \phi_i - \frac{a}{p+q} \phi_i^2 \right), i = 1, 2, h_s = H(0, \pm Y) = 0.
$$
\nIf  $\frac{g}{p} + \frac{2c}{2p+q} \phi_i - \frac{a}{p+q} \phi_i^2 = 0$ , i.e.,  $ag = \frac{2p}{p+q} c^2$ , we have  $H(\phi_1, 0) = H(0, \pm Y) = 0$ .  
\nLet  $M(\phi_1, y_i)$  be the coefficient matrix of the linearized system of (2.1) at an equilibrium point  $(\phi_1, y_i)$ ,  $J(\phi_1, y_i)$  is the corresponding Jacobi determinant of the  $M(\phi_1, y_i)$ . Then, if  $p > 0, qg > 0, ag > 0$ ,  $ag = \frac{2p}{p+q} c^2$ , we have\n
$$
J(\phi_1, 0) = -\frac{q\sqrt{c^2 + 4ag}}{2a} (c + \sqrt{c^2 + 4ag}) < 0, J(0, \pm Y) = 2pqY^2 > 0,
$$

*British Journal of Mathematics & Computer Science 4(13), 1849-1856, 2014*  
\n
$$
A_1(\phi, 0), A_2(\phi_2, 0)
$$
 and  $S_{\pm}(0,\pm Y)$ , where  
\n $\phi_1 = \frac{-c + \sqrt{c^2 + 4ag}}{-2a}, \phi_2 = \frac{c + \sqrt{c^2 + 4ag}}{2a}$  and  $Y = \sqrt{\frac{g}{p}}$ .  
\nFor the function defined by (1.9), we denote that  
\n $h_y = H(\phi_y, 0) = -\phi_1^{\frac{2g}{3}}(\frac{g}{p} + \frac{2c}{2p+q}\phi_y - \frac{a}{p+q}\phi_z^3), i = 1, 2, h_z = H(0,\pm Y) = 0.$   
\nIf  $\frac{g}{p} + \frac{2c}{2p+q}\phi_1 - \frac{a}{p+q}\phi_1^2 = 0$ , i.e.,  $ag = \frac{2p}{p+q}c^2$ , we have  $H(\phi_1, 0) = H(0,\pm Y) = 0$ .  
\nLet  $M(\phi_1, y_i)$  be the coefficient matrix of the linearized system of (2.1) at an equilibrium  
\npoint  $(\phi_1, y_i)$ ,  $J(\phi_1, y_i)$  is the corresponding Jacobi determinant of the  $M(\phi_1, y_i)$ . Then, if  
\n $p > 0, qg > 0, ag > 0, ag = \frac{2p}{p+q}c^2$ , we have  
\n $J(\phi_1, 0) = -\frac{q\sqrt{c^2 + 4ag}}{2a} (c + \sqrt{c^2 + 4ag}) < 0, J(0,\pm Y) = 2pq Y^2 > 0$ ,  
\n $J(\phi_2, 0) = -\frac{q\sqrt{c^2 + 4ag}}{2a} (c - \sqrt{c^2 + 4ag}) > 0$ ,  $Trace(M(\phi_1, 0)) = 0$ .  
\nBy the theory of planar dynamical systems [13,14], we know that for an equilibrium point  
\n $(\phi_1, y_i)$  of a planar integralbe system, if  $J < 0$  then the equilibrium point is a saddle point;  
\nif  $J > 0$  and  $Trace(M(\phi_1, 0)) = 0$  then it is a acute point; if  $J > 0$  and

By the theory of planar dynamical systems [13,14], we know that for an equilibrium point equilibrium point is zero then it is a cusp; if  $J = 0$  and the index of the equilibrium point is not zero then it is a high order equilibrium point. Using the above qualitative analysis, we can obtain the bifurcation curves and phase portraits under various parameter conditions.  $f(z) = 2pqY^2 > 0$ ,<br>  $f(z) = 2pqY^2 > 0$ ,<br>
hat for an equilibrium point<br>
rium point is a saddle point;<br>
rer point; if  $J > 0$  and<br>
and the Poincare index of the<br>
of the equilibrium point is not<br>
ratative analysis, we can obtain<br>  $= 2pqY^2 > 0,$ <br> *p*  $M(\phi_1, 0) = 0.$ <br>
for an equilibrium point<br> *p* point; if  $J > 0$  and<br>
the Poincare index of the<br> *p* equilibrium point is not<br>
we analysis, we can obtain<br>
tions.<br>
ints  $S_1$  and  $A_2$  are center<br>  $= \frac{2p$ 

Thus, the equilibrium point  $A_1$  is a saddle point, the equilibrium points  $S_{\pm}$  and  $A_2$  are center points.

 $\frac{p}{c^2}$ , we show the  $\frac{P}{q}c^2$ , we show the phase portraits of system (1.7) in Fig. 1.



#### **3 Exact Explicit Periodic Solutions of (1.4)**

In the following, we give exact explicit parametric representations of periodic solutions.

 $\frac{p}{c^2}$ .  $+q$ 

 $(1-2)$   $a < 0$ ,  $q > 0$ ,  $p < 0$ ,  $g < 0$ ,  $c < 0$ .<br> *p*  $pg > 0$ ,  $c^2 + 4ag > 0$ ,  $ag = \frac{2p}{p+q}c^2$ ,  $p \ge -2q$ .<br> **s of (1.4)**<br> **orbits** of the periodic annulus surrounding<br>  $\phi, y = \phi'$ ) be a point in the periodic orbits of<br>
plutio In this case, we have  $\phi_1 =$  $^{2}+4a\sigma$   $^{2}$  $\gamma_1$   $\gamma_2$   $\gamma_3$  $\frac{4ag}{2} = \frac{c}{2}(-1+\sqrt{\frac{9p+q}{2}})$  and the phase portraits of (  $c > 0$ . (1-2)  $a < 0$ ,  $q > 0$ ,  $p < 0$ ,  $g < 0$ ,  $c < 0$ .<br>
(1.7) for  $pg > 0$ ,  $c^2 + 4ag > 0$ ,  $ag = \frac{p}{p+q}c^2$ ,  $p \ge -2q$ .<br>
(**Solutions of (1.4)**<br>
the periodic orbits of the periodic annulus surrounding<br>
turves. Let  $(\phi, y = \phi')$  *g* < 0, *c* > 0. (1-2) *a* < 0, *q* > 0, *p* < 0, *g* < 0, *c* < 0.<br>
s of system (1.7) for  $pg > 0, c^2 + 4ag > 0, qg = \frac{2p}{p+q}c^2, p \geq -2q$ .<br> **iodic Solutions of (1.4)**<br>  $h_1$ , the periodic orbits of the periodic amulus surrou *a* b a b a g a p  $\left(\frac{1}{2}\right)a < 0, q > 0, p < 0, q < 0, e < 0.$ <br> **a** a tem (1.7) for  $pg > 0, c^2 + 4ag > 0, ag = \frac{3p}{p+q}c^2, p \ge -2q$ .<br> **Solutions of (1.4)**<br>
a periodic orbits of the periodic annulus surrounding<br>
arrows. Le  $p < 0, g < 0, c > 0.$  (1-2)  $a < 0, q > 0, p < 0, g < 0, c < 0.$ <br>
obtains of system (1.7) for  $pg > 0, c^2 + 4ag > 0, qg = \frac{2p}{p+q}e^2, p \ge -2q$ .<br> **t Periodic Solutions of (1.4)**<br>  $h \rightarrow h_1$ , the periodic orbits of the periodic amulus surroundin . (a)  $g < 0$ ,  $c > 0$ . (1-2)  $a < 0$ ,  $q > 0$ ,  $p < 0$ ,  $g < 0$ ,  $c < 0$ .<br>
Traits of system (1.7) for  $pg > 0$ ,  $e^2 + 4ag > 0$ ,  $ag = \frac{2p}{p+q}e^2$ ,  $p \ge -2q$ .<br> **Periodic Solutions of (1.4)**<br>  $h \rightarrow h_1$ , the periodic orbits of the peri 0,  $c > 0$ .  $(1-2)$   $a < 0$ ,  $q > 0$ ,  $p < 0$ ,  $g < 0$ ,  $c < 0$ .<br>
system (1.7) for  $pg > 0$ ,  $c^2 + 4ag > 0$ ,  $gg = \frac{2p}{p+q}c^2$ ,  $p \ge -2q$ .<br> **ic Solutions of (1.4)**<br>
the periodic orbits of the periodic annulus surrounding<br>
curves. Le (1.1)  $a < 0, q > 0, p < 0, q < 0, c > 0$ . (1.2)  $a < 0, q > 0, p < 0, q < 0, c < 0$ .<br>
Fig. 1 The phase portraits of system (1.7) for  $pg > 0, c^2 + 4aq > 0, qq = \frac{2p}{p^2q}c^2, p \geq -2q$ .<br> **3 Exact Explicit Periodic Solutions of (1.4)**<br> **Imma 3.1.** system (1.7) is shown in Fig. 1(1-1). Notice that  $H(A_1) = H(S_+) = 0$ , periodic orbit surrounding the center point  $A_1(\frac{\epsilon}{2\epsilon}(-1+\sqrt{\frac{P+q}{2\epsilon}}),0)$  $9p+q_{\lambda}$ 0,  $p < 0$ ,  $g < 0$ ,  $c > 0$ . (1-2)  $a < 0$ ,  $q > 0$ ,  $p < 0$ ,  $q < 0$ ,  $c < 0$ .<br>
se portraits of system (1.7) for  $pg > 0$ ,  $c^2 + 4ag > 0$ ,  $ag = \frac{2p}{p+q}c^2$ ,  $p \ge -2q$ .<br> **licit Periodic Solutions of (1.4)**<br>
When  $h \rightarrow h_1$ , the period *c b*, *p* < 0, *g* < 0, *c* > 0. (1-2) *a* < 0, *q* > 0, *p* < 0, *g* < 0, *c* < 0.<br>
hase portraits of system (1.7) for  $pg > 0$ ,  $d^2 + 4ag > 0$ ,  $ag = \frac{2p}{p+q}e^2$ ,  $p \geq -2q$ .<br> **plicit Periodic Solutions of (1.4)**<br>
When  $h$ *p* < 0, *g* < 0, *e* > 0. (1-2) *a* < 0, *q* > 0, *p* < 0, *p* < 0, *g* (*c* < 0.<br>
portraits of system (1.7) for *pg* > 0, *e*<sup>2</sup> + *ag* > 0, *ag* =  $\frac{2p}{p+q}e^2$ , *p* ≥ - 2*a*.<br> **it Periodic Solutions of (1.4)**<br>
an *h*  $z = 0$ , *g* < 0, *e* > 0. (1-2) *a* < 0, *q* > 0, *p* < 0, *g* < 0, *e* < 0.<br>
traits of system (1.7) for *pg* > 0, *e*<sup>2</sup> + *dag* > 0, *ag* =  $\frac{2p}{p+q}e^2$ , *p* ≥ −2*q*.<br> **Periodic Solutions of (1.4)**<br>  $h \rightarrow h_1$ , the p  $\frac{q}{q}$ , 0) are  $\frac{q}{q}$ ) and the phase portraits of<br>= 0, periodic orbit surrounding<br> $\frac{2}{2} \left( \frac{(p+q)^2}{(2p+q)^2} + 2 \right)$ ]. (3.1)<br>we get 0,  $g < 0, c > 0$ ,  $ag = \frac{2p}{p+q}c^2$ .<br>  $\frac{c+\sqrt{c^2+4ag}}{-2a} = \frac{c}{2a}(-1+\sqrt{\frac{9p+q}{p+q}})$  and the phase portraits of<br>
1(1-1). Notice that  $H(A_1) = H(S_{\pm}) = 0$ , periodic orbit surrounding<br>  $\sqrt{\frac{9p+q}{p+q}}$ , 0) are<br>  $\frac{2}{p+q} = \frac{a}{p$  $\frac{\overline{q}}{q}$ ) and the phase portraits of<br>= 0, periodic orbit surrounding<br> $\frac{2}{2} \left( \frac{(p+q)^2}{(2p+q)^2} + 2 \right)$ ]. (3.1)<br>we get **b**<br>
periodic orbits of the periodic annulus surrounding<br>
ex. Let  $(\phi, y = \phi')$  be a point in the periodic orbits of<br>
g wave solution is defined by  $h = h_1$ .<br>
arametric representations of periodic solutions.<br>  $> 0$ ,  $ag = \frac{2p}{$ **Solutions of (1.4)**<br>
the periodic orbits of the periodic annulus surrounding<br>
reurves. Let  $(\phi, y = \phi')$  be a point in the periodic orbits of<br>
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metric repre of system (1.7) for  $p(y > 0, e^x + 4ag > 0, ag = \frac{1}{p+q}e^x, p \ge -2q$ .<br> **dic Solutions of (1.4)**<br>  $A_1$ , the periodic orbits of the periodic annulus surrounding<br>  $A_2$  are periodic orbits of the periodic annulus surrounding<br>
trave **Solutions of (1.4)**<br>
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varyes. Let  $(\phi, y = \phi')$  be a point in the periodic orbits of<br>
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re<br>  $-\frac{c(p+q)}{a(2p+q)})^2 + \frac{c^2}{a^2} (\frac{(p+q)^2}{(2p+q)^2} + 2)].$  (3.1)<br>
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(3.1) wave solution is defined by  $h = h_1$ .<br>
ametric representations of periodic solutions.<br>
0,  $ag = \frac{2p}{p+q}c^2$ .<br>  $\frac{ag}{2a} = \frac{c}{2a}(-1+\sqrt{\frac{9p+q}{p+q}})$  and the phase portraits of<br>
e that  $H(A_1) = H(S_1) = 0$ , periodic orbit surroun entations of periodic solutions.<br>  $\frac{p}{q}c^2$ .<br>  $+\sqrt{\frac{9p+q}{p+q}}$  and the phase portraits of<br>  $= H(S_{\pm}) = 0$ , periodic orbit surrounding<br>  $\frac{q}{q}$ <br>  $\left(\frac{q}{q}\right)^2 + \frac{c^2}{a^2}\left(\frac{(p+q)^2}{(2p+q)^2} + 2\right)$ . (3.1)<br>  $\left(\frac{q}{q}\right)^2 + \$ ing wave solution is defined by  $h = h_1$ .<br>
the parametric representations of periodic solutions.<br>  $c > 0$ ,  $ag = \frac{2p}{p+q}c^2$ .<br>  $\frac{4ag}{q} = \frac{c}{2a}(-1 + \sqrt{\frac{9p+q}{p+q}})$  and the phase portraits of<br>  $\frac{d}{p}$ .<br>  $\frac{d}{p}$ .<br>  $\frac{d}{$ cit parametric representations of periodic solutions.<br>  $p, c > 0$ ,  $ag = \frac{2p}{p+q}c^2$ .<br>  $\frac{c^2+4ag}{2a} = \frac{c}{2a}(-1+\sqrt{\frac{9p+q}{p+q}})$  and the phase portraits of<br>
Notice that  $H(A_i) = H(S_{\pm}) = 0$ , periodic orbit surrounding<br>  $\frac{4}{\sqrt$ (v, y, - y y oc a point in the periodic solution is<br>solution is defined by  $h = h_1$ .<br><br>  $g = \frac{2p}{p+q}e^2$ .<br>  $\frac{c}{2a}(-1+\sqrt{\frac{9p+q}{p+q}})$  and the phase portraits of<br>  $H(A_1) = H(S_+) = 0$ , periodic orbit surrounding<br>
(e)<br>  $\frac{c(p+q)}$ ing wave solution is defined by  $h = h_1$ .<br>
parametric representations of periodic solutions.<br>  $c > 0$ ,  $ag = \frac{2p}{p+q}c^2$ .<br>  $+\frac{4ag}{q} = \frac{c}{2a}(-1+\sqrt{\frac{9p+q}{p+q}})$  and the phase portraits of<br>
totice that  $H(A_1) = H(S_1) = 0$ , perio

$$
y^{2} = \frac{a}{p+q} \left[ -( \phi - \frac{c(p+q)}{a(2p+q)})^{2} + \frac{c^{2}}{a^{2}} \left( \frac{(p+q)^{2}}{(2p+q)^{2}} + 2 \right) \right].
$$
 (3.1)

Substituting  $(3.1)$  into the first equation of  $(1.7)$  and integrating it, we get

$$
d\xi = \frac{d\phi}{\sqrt{\frac{a}{p+q}\left[D^2 - (\phi - \frac{c(p+q)}{a(2p+q)})^2\right]}}
$$

that is

*British Journal of Mathematics* & Computer Science 4(13), 1849-1856, 2014  
that is  

$$
\xi = \sqrt{\frac{a}{p+q}} \int_{\phi}^{\phi} \frac{d\phi}{\sqrt{D^2 - (\phi - \frac{c(p+q)}{a(2p+q)})^2}}, \qquad (3.2)
$$
  
where 
$$
D^2 = \frac{c^2}{a^2} \left(\frac{(p+q)^2}{(2p+q)^2} + 2\right), B = -\frac{c(p+q)}{a(2p+q)}, \text{ which implies the following parametric representations:}
$$

$$
\phi(\xi) = \frac{c}{2a} \left(-1 + \sqrt{\frac{9p+q}{p+q}}\right) - D \sin\left(\sqrt{\frac{a}{p+q}}\right). \qquad (3.3)
$$
  
(3.3) gives a periodic solution and the profile is shown in Fig. 2(2-1).  
**Remark.** To the best of our knowledge, the solution (3:3) of Eq. (1.4) has not been reported in literature.  
**Case (II).**  $a < 0, q > 0, p < 0, g < 0, c < 0, ag = \frac{2p}{p+q}c^2$ .  
In this case, we have the phase portrait of system (1.7) shown in Fig. 1(1-2). Paralleled to the classes (1), system (1.7) has a parametric representation of the periodic orbit as (3.3). It gives another periodic solution and the profile is shown in Fig. 2 (2-2).

where  $D^2 = -$ 2 ( $(\nu + 4)$  ( $\gamma$ )  $+q)^2$  (2)  $R$   $c(p+q)$  (1)  $R$   $(1-p)$  $+(q)^2$   $a(2p+q)$ , which implies the 1  $+q$ )  $\qquad \qquad$  $\frac{q}{q}$ , which implies the following parametric  $+q$ )

representations:

$$
\phi(\xi) = \frac{c}{2a}(-1 + \sqrt{\frac{9p+q}{p+q}}) - D\sin(\sqrt{\frac{a}{p+q}}\xi).
$$
 (3.3)

(3.3) gives a periodic solution and the profile is shown in Fig. 2(2-1).

**Remark.** To the best of our knowledge, the solution (3*:*3) of Eq. (1.4) has not been reported in literature.

Case (II). 
$$
a < 0, q > 0, p < 0, g < 0, c < 0, ag = \frac{2p}{p+q}c^2
$$
.

In this case, we have the phase portrait of system  $(1.7)$  shown in Fig.  $1(1-2)$ . Paralleled to the Cases (I), system (1.7) has a parametric representation of the periodic orbit as (3.3). It gives another periodic solution and the profile is shown in Fig. 2 (2-2).



(2-1)  $a = g = -1, c = q = 1, p = -2.$  $(2-2)$   $a = g = c = -1, q = 1, p = -2.$ Fig. 2 Periodic solutions of Eq.(1.1) for  $pg > 0, c^2 + 4ag > 0, ag = \frac{2pc^2}{p+q}, p \ge -2q$ .

#### **4 Conclusion**

In this paper, we used the qualitative analysis methods of dynamical systems to investigate the periodic solutions of ZK (2, 4, -2) equation. As a result, we obtained two of new exact periodic solutions. The phase portrait bifurcation of the travelling wave system corresponding to the equation is given. The graph of the solutions are given with the numerical simulation. The phase portrait bifurcation of the travelling wave system corresponding to the equation is given. The graph of the solutions are given with the numerical simulation. Based on the ideas of finding limit cycles by Abelian integral, see [19,20], we will also investigate the isolated travelling waves of the system in the next paper.

#### **Competing Interests**

Authors have declared that no competing interests exist.

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