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Periodic Travelling Wave Solutions of ZK (2, 4, -2) Equation

Minzhi Wei[1*] and Sumin Yang[2]

[1]*School of Information and Statistics, Guangxi University of Finance and Economics Nanning, Guangxi, 530003, P. R. China.*
[2]*Guangxi Technological College of Machinery and Electricity Nanning, Guangxi, 530007, P. R. China.*

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Abstract

In this paper, the qualitative analysis methods of dynamical systems are used to investigate the periodic travelling wave solutions of ZK (2, 4,-2) equation. The phase portrait bifurcation of the travelling wave system corresponding to the equation is given. The explicit expressions of the periodic travelling wave solutions are obtained by using the portraits. The graph of the solutions are given with the numerical simulation.

Keywords: Periodic travelling wave solutions, ZK (2, 4, -2) equation, bifurcation method.
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1 Introduction

The study of travelling wave solutions in particular, solitons, of Partial Deffierential Equations (what called PDEs), for various nonlinear evolution equations in mathematical physics plays an important role in soliton theory. To obtain the travelling wave solutions for PDEs, a lot of systematic methods have been developed for soliton equations, such as the inverse scattering method, the Backlund and the Darboux transformations, the tanh-function method, the homogeneous balance method, the extended tanh-function method and others [1-7,18].

The ZK equation governs the behavior of weakly nonlinear ion-acoustic waves in aplasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [8]. The ZK equation, which is a more isotropic two-dimensional, was first derived for describing weakly nonlinear ion-acoustic waves in a strongly magnetized lossless plasma in two-dimension [9].

Recently, the following ZK equation

Corresponding author: weiminzhi@21cn.com;

$$u_t + auu_x + b(u_{xx} + u_{yy})_x = 0, \tag{1.1}$$

was investigated by BK. Shivamoggi [10], A. M. Wazwaz [11] and VE. Zakharov etc [9] with various distinct approaches.

More recently, by using the sine-cosine method and the tanh method, A. M. Wazwaz [12] investigated the ZK (n, -n, 2n) equation,

$$u_t + a(u^n)_x + b[u^{-n}(u^{2n})_{xx} + k(u^n)_{yy}]_x = 0, \tag{1.2}$$

and the ZK (n,2n,-n) equation,

$$u_t + a(u^n)_x + b[u^{2n}(u^{-n})_{xx} + k(u^n)_{yy}]_x = 0, \tag{1.3}$$

Where a, b, k are three non-zero real numbers, and obtained a family of solutions:

$$u_1 = \begin{cases} \dfrac{2nc(b+k)}{a[b(3n-1)+k(n+1)]}\sin^2[\sqrt{\dfrac{n-1}{2n}}(x+y-ct)]^{\frac{1}{n-1}}, |\mu\xi| < \pi, \\ 0, otherwise, \end{cases}$$

$$u_2 = \begin{cases} \dfrac{2nc(b+k)}{a[b(3n-1)+k(n+1)]}\cos^2[\sqrt{\dfrac{n-1}{2n}}(x+y-ct)]^{\frac{1}{n-1}}, |\mu\xi| < \pi, \\ 0, otherwise, \end{cases}$$

other exact explicit solutions were listed in [13]. However, the bifurcation behavior of the travelling wave solutions for corresponding travelling wave equations haven't studied in its parameter space. It is very important to understand the dynamical behavior of solutions for the travelling wave equations, the related investigation of (1.2) and (1.3) hasn't been mentioned in the literatures. In this paper, we shall continue to study and obtain the periodic travelling wave solutions of Eq. (1.3) for $n = 2$ (simply called $ZK(2,4,2)$ equation), which employs bifurcation method of dynamical systems [13,14] to investigate the following equation:

$$u_t + a(u^2)_x + b[u^4(u^{-2})_{xx} + k(u^2)_{yy}]_x = 0, \tag{1.4}$$

Taking the transformation $u(x, y, t) = \phi(x + y - ct) = \phi(\xi)$, where c is the wave speed, then (1.4) becomes

$$-c\phi' + a(u^2)' + b[\phi^4(\phi^{-2})'' + k(u^2)'']' = 0, \tag{1.5}$$

where "`" is the derivative with respect to ξ. Taking the integration once on both sides leads to

$$-c\phi + a(u^2)' + b[\phi^4(\phi^{-2})'' + k(u^2)''] = g, \tag{1.6}$$

where g is the integration constant. Clearly, (1.6) is equivalent to the following two-dimensional system

$$\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = y, \quad \frac{\mathrm{dy}}{\mathrm{d}\xi} = \frac{g + c\phi - a\phi^2 - py^2}{q\phi}, \tag{1.7}$$

where $p = 6b + 2k, q = 2k - 2b,$ which has the first integral

$$y^2 = \phi^{-\frac{2p}{q}}\left(\frac{g}{p}\phi^{\frac{2p}{q}} + \frac{2c}{2p+q}\phi^{\frac{2p}{q}+1} - \frac{a}{p+q}\phi^{\frac{2p}{q}+2} + h\right) \tag{1.8}$$

Or

$$H(\phi, y) = \phi^{\frac{2p}{q}} y^2 - \phi^{\frac{2p}{q}}\left(\frac{g}{p} + \frac{2c}{2p+q}\phi - \frac{a}{p+q}\phi^2\right) = h\,. \tag{1.9}$$

Obviously system (1.7) is a five-parameter planar dynamical system depending on the parameter group (g, c, a, p, q). Since the phase orbits defined by the vector fields of Eq. (1.7) determine all traveling wave solutions of system (1.3), we should investigate the bifurcations of phase portraits of system (1.7) in the phase plane (ϕ, y) as the parameters g, c, a, p and q are changed.

The rest of this paper is organized as follows: in Section 2, we discuss the bifurcations of phase portraits of system (1.7), where explicit parametric conditions will be derived. In Section 3, we give exact explicit parametric representations for periodic solutions of Eq. (1.4) for $n = 2$. Section 4 contains the concluding remarks.

2 Bifurcations and Phase Portraits of System (1.7)

In this section, we discuss the existence of periodic solutions of (1.3) by the bifurcation method [14-17]. System (1.7) has the same phase orbits as the following system

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = q\phi y, \ \frac{\mathrm{dy}}{\mathrm{d}\tau} = g + c\phi - a\phi^2 - py^2, \tag{2.1}$$

except for the straight line $\phi = 0,$ where $\mathrm{d}\xi = q\phi \mathrm{d}\tau$.

If $pg > 0, c^2 + 4ag > 0,$ system (1.7) has four equilibrium points:

$A_1(\phi_1, 0), A_2(\phi_2, 0)$ and $S_\pm(0, \pm Y)$, where

$$\phi_1 = \frac{-c + \sqrt{c^2 + 4ag}}{-2a}, \phi_2 = \frac{c + \sqrt{c^2 + 4ag}}{2a} \text{ and } Y = \sqrt{\frac{g}{p}}.$$

For the function defined by (1.9), we denote that

$$h_i = H(\phi_i, 0) = -\phi_i^{\frac{2p}{q}} (\frac{g}{p} + \frac{2c}{2p+q}\phi_i - \frac{a}{p+q}\phi_i^2), i = 1, 2, \ h_s = H(0, \pm Y) = 0.$$

If $\frac{g}{p} + \frac{2c}{2p+q}\phi_1 - \frac{a}{p+q}\phi_1^2 = 0$, i.e., $ag = \frac{2p}{p+q}c^2$, we have $H(\phi_1, 0) = H(0, \pm Y) = 0.$

Let $M(\phi_i, y_i)$ be the coeficient matrix of the linearized system of (2.1) at an equilibrium point (ϕ_i, y_i), $J(\phi_i, y_i)$ is the corresponding Jacobi determinant of the $M(\phi_i, y_i)$ .Then, if $p > 0, qg > 0, ag > 0, \ ag = \frac{2p}{p+q}c^2$, we have

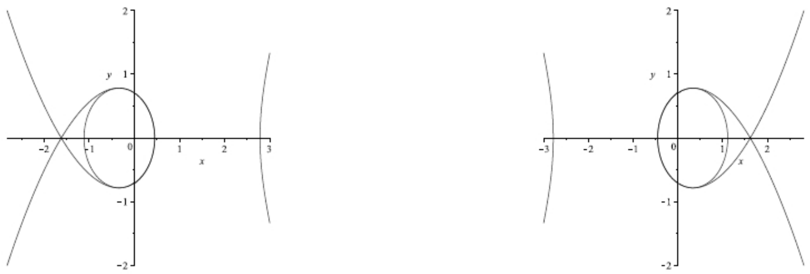
$$J(\phi_1, 0) = -\frac{q\sqrt{c^2 + 4ag}}{2a}(c + \sqrt{c^2 + 4ag}) < 0, J(0, \pm Y) = 2pqY^2 > 0,$$

$$J(\phi_2, 0) = -\frac{q\sqrt{c^2 + 4ag}}{2a}(c - \sqrt{c^2 + 4ag}) > 0, Trace(M(\phi_1, 0)) = 0.$$

By the theory of planar dynamical systems [13,14], we know that for an equilibrium point (ϕ_i, y_i) of a planar integrable system, if $J < 0$ then the equilibrium point is a saddle point; if $J > 0$ and $Trace(M(\phi_1, 0)) = 0$ then it is a center point; if $J > 0$ and $(Trace(M(\phi_1, 0)))^2 - 4J(\phi_i, y_i) > 0$ then it is a node; if $J = 0$ and the Poincar*e* index of the equilibrium point is zero then it is a cusp; if $J = 0$ and the index of the equilibrium point is not zero then it is a high order equilibrium point. Using the above qualitative analysis, we can obtain the bifurcation curves and phase portraits under various parameter conditions.

Thus, the equilibrium point A_1 is a saddle point, the equilibrium points $S_\pm$ and A_2 are center points.

For our purpose, in the parameter region: $pg > 0, c^2 + 4ag > 0, \ ag = \frac{2p}{p+q}c^2$, we show the phase portraits of system (1.7) in Fig. 1.



(1-1) $a < 0,\ q > 0,\ p < 0,\ g < 0,\ c > 0.$ (1-2) $a < 0,\ q > 0,\ p < 0,\ g < 0,\ c < 0.$

Fig. 1 The phase portraits of system (1.7) for $pg > 0, c^2 + 4ag > 0, ag = \frac{2p}{p+q}c^2, p \geq -2q.$

3 Exact Explicit Periodic Solutions of (1.4)

Lemma 3.1. When $h \to h_1$, the periodic orbits of the periodic annulus surrounding $(\phi_1, 0)$ approach to the boundary curves. Let $(\phi, y = \phi')$ be a point in the periodic orbits of system (1.6). Then, the periodic travelling wave solution is defined by $h = h_1$.

In the following, we give exact explicit parametric representations of periodic solutions.

Case (I). $a < 0, q > 0, p < 0, g < 0, c > 0,\ ag = \dfrac{2p}{p+q}c^2.$

In this case, we have $\phi_1 = \dfrac{-c + \sqrt{c^2 + 4ag}}{-2a} = \dfrac{c}{2a}(-1 + \sqrt{\dfrac{9p+q}{p+q}})$ and the phase portraits of system (1.7) is shown in Fig. 1(1-1). Notice that $H(A_1) = H(S_\pm) = 0,$ periodic orbit surrounding the center point $A_1(\dfrac{c}{2a}(-1 + \sqrt{\dfrac{9p+q}{p+q}}), 0)$ are

$$y^2 = \frac{a}{p+q}[-(\phi - \frac{c(p+q)}{a(2p+q)})^2 + \frac{c^2}{a^2}(\frac{(p+q)^2}{(2p+q)^2} + 2)]. \tag{3.1}$$

Substituting (3.1) into the first equation of (1.7) and integrating it, we get

$$\mathrm{d}\xi = \frac{\mathrm{d}\phi}{\sqrt{\dfrac{a}{p+q}[D^2 - (\phi - \dfrac{c(p+q)}{a(2p+q)})^2]}}$$

that is

$$\xi = \sqrt{\frac{a}{p+q}}\int_{\phi_1}^{\phi} \frac{\mathrm{d}\phi}{\sqrt{D^2 - (\phi - \frac{c(p+q)}{a(2p+q)})^2}}, \tag{3.2}$$

where $D^2 = \frac{c^2}{a^2}(\frac{(p+q)^2}{(2p+q)^2} + 2),\ B = -\frac{c(p+q)}{a(2p+q)}$, which implies the following parametric representations:

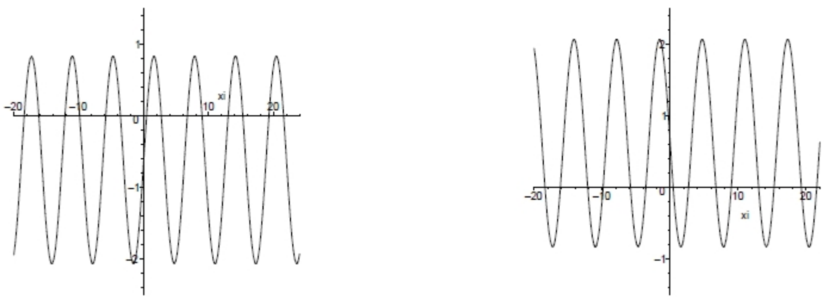
$$\phi(\xi) = \frac{c}{2a}(-1 + \sqrt{\frac{9p+q}{p+q}}) - D\sin(\sqrt{\frac{a}{p+q}}\xi). \tag{3.3}$$

(3.3) gives a periodic solution and the profile is shown in Fig. 2(2-1).

Remark. To the best of our knowledge, the solution (3.3) of Eq. (1.4) has not been reported in literature.

Case (II). $a < 0, q > 0, p < 0, g < 0, c < 0,\ ag = \frac{2p}{p+q}c^2.$

In this case, we have the phase portrait of system (1.7) shown in Fig. 1(1-2). Paralleled to the Cases (I), system (1.7) has a parametric representation of the periodic orbit as (3.3). It gives another periodic solution and the profile is shown in Fig. 2 (2-2).



(2-1) $a = g = -1, c = q = 1, p = -2.$ (2-2) $a = g = c = -1, q = 1, p = -2.$

Fig. 2 Periodic solutions of Eq.(1.1) for $pg > 0, c^2 + 4ag > 0, ag = \frac{2pc^2}{p+q}, p \geq -2q.$

4 Conclusion

In this paper, we used the qualitative analysis methods of dynamical systems to investigate the periodic solutions of ZK (2, 4, -2) equation. As a result, we obtained two of new exact periodic solutions. The phase portrait bifurcation of the travelling wave system corresponding to the equation is given. The graph of the solutions are given with the numerical simulation. The phase portrait bifurcation of the travelling wave system corresponding to the equation is given. The graph of the solutions are given with the numerical simulation. Based on the ideas of finding limit cycles by Abelian integral, see [19,20], we will also investigate the isolated travelling waves of the system in the next paper.

Competing Interests

Authors have declared that no competing interests exist.

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