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New Optimal Family of Iterative Methods for Solving Nonlinear Equations

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Abstract

Motivated by the results of J. Kou, et al. 2007 [1] which is a family of iterative methods having third order of convergence in general and optimal fourth order of convergence for a particlue case, in this paper, a new class of optimal fourth-order iterative methods is obtained based on that and by using the weight function approach. The convergence analysis of our new class of iterative methods is presented in this paper. Moreover, several numerical examples are considered and compared with the available methods in the literature to confirm our theoretical results.

Keywords: Newton's method; iterative methods; weight function; efficiency index; order of convergence; optimal family.

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1 Introduction

A well-known and a powerful method for finding a simple zero of a nonlinear equation

$$f(x) = 0, \tag{1.1}$$

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is the Newton's method which is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1.2}$$

This method has a quadratic convergence, where $f : I \subseteq R \to R$ is a real and sufficiently smooth function in a real open interval I with the assumption that f has a simple zero at α in I, i.e., $f(\alpha) = 0$ and $f'(\alpha) \neq 0$.

To improve the local order of convergence, many modified fourth-order iterative methods are proposed using different techniques, see, for examples, IA. Al-Subaihi 2014 [2], IA. Al-Subaihi, Hani I. Siyyam 2014 [3], IA. Al-Subaihi, Awatif J. Alqarni 2014 [4], C. Chun 2008 [5], 2008 [6], 2007 [7], Chun and Yoon 2008 [8], MA. Hafiz 2014 [9], JP. Jaiswal 2014 [10], Hani I. Siyyam, IA. Al-Subaihi 2013 [11] and references therein. A family of iterative methods is developed by J. Kou, et al. 2007 [1] as:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= x_n - \theta \frac{f(x_n) + f(y_n)}{f'(x_n)} - (1-\theta)\frac{(f(x_n))^2}{f'(x_n)(f(x_n) - f(y_n))}, \end{aligned} \tag{1.3}$$

where θ is any real number. The error equation corresponding to (1.3) is as follows:

$$\begin{aligned} e_{n+1} &= x_{n+1} - \alpha \\ &= (\theta c_2^2 + c_2^2)e_n^3 + (4\theta c_2 c_3 + 3c_2 c_3 - 6\theta c_2^3 - 3c_2^3)e_n^4 + O(e_n^5). \end{aligned} \tag{1.4}$$

From equation (1.4), we can see that the order of convergence of the family (1.3) is three in the case of $\theta \neq -1$ and four in the case of $\theta = -1$.

Per one cycle, any member in the family (1.3) requires two evaluations of the function and one evaluation of its first derivative, so its efficiency index is $3^{1/3} \approx 1.422$ in the case of $\theta \neq -1$ and $4^{1/3} \approx 1.5874$ in the case of $\theta = -1$ which is optimal only in the last case, where the efficiency index of a method is defined to be $\phi^{1/\beta}$, where ϕ is the order of convergence and β is the total number of the function and its derivative evaluations.

The construction of the new fourth-order class of optimal family of iterative methods with the convergence analysis will be presented in next section. Several numerical examples are considered in Section 3 and compared with the available methods in the literature to confirm the efficiency and the accuracy of the proposed family of iterative methods. Finally, some conclusions are presented.

2 Construction of the New Optimal Family of Iterative methods

To construct the new optimal family of iterative methods with fourth-order of convergence for solving nonlinear equation (1.1), consider the two-step family of iterative methods of J. Kou, et al. 2007 [1]. From equation (1.4), the order of convergence of this family is three in the case of $\theta \neq -1$ and reaches to four in the case of $\theta = -1$ and using three pieces of information per one cycle. We make our new optimal family of iterative methods based on (1.3) and also make use of the weight function approach. Therefore, to get an optimal family of fourth-order iterative methods, we suggest the following without memory iteration

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= x_n - \left[\theta\frac{f(x_n)+f(y_n)}{f'(x_n)} + (1-\theta)\frac{(f(x_n))^2}{f'(x_n)(f(x_n)-f(y_n))}\right] \\ &\quad \times H(v), \end{aligned} \tag{2.1}$$

where $H(v)$ is a real valued weight function with $v = \frac{f(y_n)}{f(x_n)}$. The next theorem illustrates that the order of convergence of the new family of iterative methods (2.1) is four for any $\theta \in R$ under certain conditions on the weight function $H(v)$. Per one cycle, any member of this family requires two evaluations of the function and one evaluation of its first derivative so its efficiency index is $4^{1/3} \approx 1.5874$, which is optimal according to Kung and Traub conjecture HT. Kung, JF. Traub 1974 [12].

Theorem 2.1. *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \subseteq R \to R$ for an open interval I. If x_0 is sufficiently close to α, then the family of iterative methods (2.1) has fourth-order of convergence when $H(0) = 1$, $H'(0) = 0$, $H''(0) = 2(\theta + 1)$ and $|H'''(0)| < \infty$.*

Proof. Let α be a simple zero of equation (1.1) and $x_n = \alpha + e_n$. By Taylor expansion, we have

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + O(e_n^6)], \tag{2.2}$$

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + O(e_n^5)], \tag{2.3}$$

where $c_k = \frac{f^{(k)}(\alpha)}{k!f'(\alpha)}, k = 2, 3, \dots$.
Dividing (2.2) by (2.3), gives us

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + (-2c_3 + 2c_2^2)e_n^3 + (-3c_4 + 7c_2c_3 - 4c_2^3)e_n^4 + O(e_n^5).$$

Substituting the last equation into y_n in (2.1), we have:

$$y_n = \alpha + c_2e_n^2 + (+2c_3 - 2c_2^2)e_n^3 + (+3c_4 - 7c_2c_3 + 4c_2^3)e_n^4 + O(e_n^5).$$

Expanding $f(y_n)$ about α to get

$$f(y_n) = f'(\alpha)[c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 5c_2^3)e_n^4 + O(e_n^5)]. \tag{2.4}$$

Dividing (2.4) by (2.2), we have

$$v = c_2e + (2c_3 - 3c_2^2)e^2 + (3c_4 - 10c_2c_3 + 8c_2^3)e^3 + (4c_5 - 14c_2c_4 - 8c_3^2 + 37c_3c_2^2 - 20c_2^4)e^4 + O(e_n^5).$$

Expanding $H(v)$ by Taylor's polynomial of the second order at the point $v = 0$,

$$\begin{aligned} H(v) &= H(0) + H'(0) + \frac{1}{2}H''(0)v^2, \\ &= 1 + (\theta+1)v^2, \\ &= 1 + (\theta c_2^2 + c_2^2)e^2 + (4\theta c_2c_3 - 6\theta c_2^3 + 4c_2c_3 - 6c_2^3)e^3 + (6\theta c_2c_4 - 32\theta c_3c_2^2 + 25\theta c_2^4 + 4\theta c_3^2 \\ &\quad + 6c_2c_4 - 32c_3c_2^2 + 25c_2^4 + 4c_3^2)e^4 + O(e_n^5). \end{aligned} \tag{2.5}$$

Now substituting $(2.2) - (2.5)$ into x_{n+1} in (2.1), to get

$$x_{n+1} = \alpha + (-c_2c_3 + 3c_2^3)e^4 + O(e_n^5). \tag{2.6}$$

Therefore,

$$\begin{aligned} e_{n+1} &= x_{n+1} - \alpha \\ &= (-c_2c_3 + 3c_2^3)e^4 + O(e_n^5). \end{aligned} \tag{2.7}$$

Which indicates that the order of convergence of the family (2.1) is exactly four for any value of θ. This completes the proof. □

As we see from the last theorem that the choice of the weight function is so important to make the order of the proposed class of iterative methods optimal. A typical form of the real-valued weight function $H(v)$ is

$$H(v) = 1 + (1+\theta)v^2 + dv^3, d \in R. \tag{2.8}$$

3 Numerical Examples

In this section we check the effectiveness of the new developed class of iterative methods defined in (2.1). For the purpose of comparison, we specifically take $\theta = 0$ in (2.1) and $d = 0$ in $H(v)$, (2.8), then the fourth order method is denoted by (SM1) and can be expressed as:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= x_n - [\frac{(f(x_n))^2}{f'(x_n)(f(x_n) - f(y_n))}][1 + (\frac{f(y_n)}{f(x_n)})^2 + (\frac{f(y_n)}{f(x_n)})^3]. \end{aligned} \tag{3.1}$$

Take $\theta = -1$ in (2.1) and $d = 1$ in $H(v)$, (2.8), then the fourth order method is denoted by (SM2) and can be expressed as:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= x_n + [\frac{f(x_n) + f(y_n)}{f'(x_n)} - 2\frac{(f(x_n))^2}{f'(x_n)(f(x_n) - f(y_n))}][1 + (\frac{f(y_n)}{f(x_n)})^2 + (\frac{f(y_n)}{f(x_n)})^3], \end{aligned} \tag{3.2}$$

comparing with the original Kou's family of iterative methods defined in (1.3), specifically, when $\theta = 0$, then the third order method is denoted by (KM1) and can be expressed as:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= x_n - [\frac{(f(x_n))^2}{f'(x_n)(f(x_n) - f(y_n))}], \end{aligned} \tag{3.3}$$

and when $\theta = -1$, then the fourth order method is denoted by (KM2) and can be expressed as:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= x_n + [\frac{f(x_n) + f(y_n)}{f'(x_n)} - 2\frac{(f(x_n))^2}{f'(x_n)(f(x_n) - f(y_n))}], \end{aligned} \tag{3.4}$$

and the fourth order method (KM3) of R. King 1973 [13], which is given by:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2)f(y_n)}\frac{f(y_n)}{f'(x_n)}, \tag{3.5}$$

with $\beta = 0$, the fourth order method (ESM) of R. Ezzati, F. Saleki 2011 [14], which is given by:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= y_n + \frac{f(y_n)}{f'(x_n)} - 2\frac{f(x_n)f(y_n)}{f'(x_n)(f(x_n)-f(y_n))}. \end{aligned} \tag{3.6}$$

The test problems and their zeros found up to the 28 decimal places are as follows:

Example	the approximate zero α
$f_1(x) = x^3 + 4x^2 - 10,$	1.3652300134140968457608068289,
$f_2(x) = \sin^2 x - x^2 + 1,$	1.4044916482153412260350868178,
$f_3(x) = ln(x^2 + x + 2) - x + 1,$	4.1525907367571582749969890004,
$f_4(x) = \tan^{-1}(x),$	0.0,
$f_5(x) = 10xe^{-x^2} - 1,$	1.6796306104284499406749203389,
$f_6(x) = \cos(\frac{\pi x}{2}) + x^2 - \pi,$	2.0347248962791266103514465112.

All computations were done using MATLAB (R2011a) with 1000 digit floating arithmetic. The following criteria

$$|x_{n+1} - x_n| < \varepsilon \quad and \quad |f(x_n)| < \varepsilon,$$

are used for stopping computer programmes. Displayed in Tables 1 are the number of iterations (IT) such that the stopping criteria satisfied, where ε is taken to be 10^{-250}, the number of functional evaluations (NFE) and the computational order of convergence (COC) which can be approximated using the formula

$$COC \approx \frac{\ln|(x_n - x_{n-1})/(x_{n-1} - x_{n-2})|}{\ln|(x_{n-1} - x_{n-2})/(x_{n-2} - x_{n-3})|}.$$

In Table 2, the values of $|f(x_n)|$ when the number of functional evaluations (NFE) is taken to be fixed and equal to 12. Moreover, displayed are the distance of two consecutive approximations $|(x_{n+1} - x_n)|$, the distance between x_{n+1} and the zero and the computational order of convergence (COC), where x_n is taken to be the approximate value of the exact zero. The starting point is chosen a random integer near the exact root. The sets of starting values from which the iteration will converge to each root are recently considered and known as basins of attraction, see for examples, B. Neta, et al. 2012 [15], M. Basto 2013 [16].

Table 1: Comparison of IT, NFE and COC of various fourth order iterative methods and the new method (SM1), (SM2).

	SM1		SM2		KM1		KM2		KM3		ESM	
	IT	NFE	IT	NFE	IT	NFE	IT	NFE	IT	NFE	IT	NFE
$f_1(x)$	5	15	6	18	6	18	5	15	5	15	5	15
$f_2(x)$	6	18	6	18	7	21	6	18	5	15	5	15
$f_3(x)$	4	12	4	12	5	15	4	12	4	12	4	12
$f_4(x)$	5	15	5	15	5	15	6	18	4	12	6	18
$f_5(x)$	6	18	6	18	7	21	6	18	5	15	5	15
$f_6(x)$	6	18	6	18	7	21	6	18	6	18	5	15

Table 2: Numerical results with fixed NFE = 12.

		$\lvert f(x_n)\rvert$	$\lvert x_{n+1}-x_n\rvert$	$\lvert x_{n+1}-\alpha\rvert$	COC
$f_1(x), x_0 = 2.0$	SM1	3.47129e-136	1.00511e-34	2.10211e-137	4.0
	SM2	2.23992e-136	9.00843e-35	1.35642e-137	4.0
	KM1	7.81393e-48	1.25333e-16	4.73187e-49	3.0
	KM2	3.08659e-126	2.75642e-32	1.86914e-127	4.0
	KM3	3.66626e-162	3.98384e-41	2.22017e-163	4.0
	ESM	3.08659e-126	2.75644e-32	1.86914e-127	4.0
$f_2(x), x_0 = 1.0$	SM1	0.21834e-145	0.31490e-36	0.87912e-146	4.0
	SM2	0.55774e-140	0.70792e-35	0.22442e-140	4.0
	KM1	0.13824e-42	0.44878e-14	0.55553e-43	3.0
	KM2	0.16223e-82	0.14752e-20	0.65165e-83	4.0
	KM3	5.34405e-147	2.68794e-37	2.15271e-147	4.0
	ESM	1.61618e-83	1.47522e-21	6.51037e-84	4.0
$f_3(x), x_0 = 2.0$	SM1	2.58556e-468	8.46144e-117	4.29291e-468	4.0
	SM2	2.58108e-468	8.45778e-117	4.28548e-468	4.0
	KM1	1.04901e-163	3.62976e-54	1.74171e-163	3.0
	KM2	2.75034e-459	1.44161e-114	4.56652e-459	4.0
	KM3	1.52734e-134	2.33039e-27	1.52734e-134	4.0
	ESM	2.75034e-459	1.44161e-114	4.56635e-459	4.0
$f_4(x), x_0 = -1.0$	SM1	4.87354e-154	2.93908e-31	4.87354e-154	5.0
	SM2	1.05014e-206	8.60779e-42	1.05014e-206	5.0
	KM1	1.67177e-179	2.37289e-36	1.67177e-179	5.0
	KM2	1.03426e-74	2.15563e-15	1.03426e-74	5.0
	KM3	1.52734e-134	2.33039e-27	1.52734e-134	5.0
	ESM	1.03426e-74	2.1556e-15	1.03426e-74	5.0
$f_5(x), x_0 = 1.0$	SM1	0.11363e-107	0.70307e-27	0.41124e-108	4.005
	SM2	0.11434e-107	0.70364e-27	0.41213e-108	4.005
	KM1	0.12924e-48	0.37082e-16	0.46654e-49	3.0
	KM2	0.16154e-103	0.69153e-26	0.58425e-104	4.0
	KM3	1.35035e-115	1.56891e-29	4.88568e-116	4.0
	ESM	1.61426e-104	6.91533e-27	5.84052e-105	4.0
$f_6(x), x_0 = 2.0$	SM1	0.32565e-127	0.39032e-31	0.85974e-128	3.996
	SM2	0.82592e-126	0.87648e-31	0.21825e-126	3.996
	KM1	0.31894e-52	0.98771e-17	0.84252e-53	3.0
	KM2	0.85856e-157	0.16753e-38	0.22756e-157	4.0
	KM3	9.01674e-93	2.69603e-23	2.38622e-93	4.0
	ESM	8.58479e-158	1.67533e-39	2.27191e-158	4.0

4 Conclusions

In this paper a weighted optimal family extension of J. King, et al. 2007 [1] for solving nonlinear equations is presented and analyzed. Each member in the developed family is fourth order iterative method. The efficiency index of the family is $4^{1/3} \approx 1.5874$. Therefore, this iterative methods supports the conjecture of HT. Kung, JF. Trau 1974 [12]. To illustrate the performance and the accuracy of our family several numerical examples are presented and compared to different fourth order iterative methods. From Table 2, it can be seen that our results are accurate results.

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Competing Interests

The author declares that no competing interests exist.

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