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On Parametric Generalization of 'Useful' R-norm Information Measure

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Abstract

In the literature of information measure, there exist many well known parametric generalized information measures with their merits and limitations. In the present paper a 'useful' R-norm information measure of type and degree is introduce and characterized axiomatically. This new measure is parametric generalization of 'useful' R-norm information measures introduced and characterized by the authors earlier refer to Hooda et al. [11]. Properties of the new generalized 'useful' R-norm information measure of type and degree have also been studied. The new measure has been applied in studying the lower and upper bounds of a generalized 'useful' R-norm mean codeword length.

Keywords: R-norm information; mean codeword length; Kraft's inequality; code alphabets and Holder's inequality.

1 Introduction

We consider the set of positive real numbers, not equal to 1 denoted by $\Re^{+} = \{R : R > 0 \neq 1\}$. Let Δ_n with $n \geq 2$ be the set of all probability distributions

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$$P = \left\{ (p_1, p_2, \ldots, p_n), p_i \geq 0 \text{ and} \sum_{i=1}^{n} p_i = 1 \right\}.$$

Boekee and Lubbe [1] considered R-norm information of distribution P defined for $R \in \Re^+$ by

$$H_R(P) = \frac{R}{R-1}\left[1 - \left(\sum_{i=1}^{n} p_i^R\right)^{\frac{1}{R}}\right],\ R(>0) \neq 1. \tag{1.1}$$

The R-norm information measure (1.1) is a real functions $\Delta_n \to \Re^+$, where $n \geq 2$ and $\Re^+$ is the set of positive real numbers. This measure is different from entropies of Shannon [2], Renyi [3], Havrda and Charvat [4], and Daroczy [5]. The most interesting property is that in case $R \to 1$, R-norm information measure approaches to Shannon's entropy [2] and when $R \to \infty$, $H_R(P) \to 1 - \max p_i,\ \forall\ i = 1,2,\ldots,n$.

The measure (1.1) can be generalized in so many ways; Hooda and Ram [6] defined the following journalized 'useful' information measure:

$$H_R^{\beta}(P) = \frac{R}{R+\beta-2}\left[1 - \left(\sum_{i=1}^{n} p_i^{\frac{R}{2-\beta}}\right)^{\frac{2-\beta}{R}}\right], 0 < \beta \leq 1, \quad R(>0) \neq 1. \tag{1.2}$$

The measure given by (1.2) was called generalized R-norm information measure of degree β as it reduces to (1.1) when $\beta = 1$.

Hooda and Sharma [7] introduced and characterized parametric generalization of (1.2) as given below:

$$H_R^{\alpha,\beta}(P) = \frac{R}{R+\beta-2\alpha}\left[1 - \left(\sum_{i=1}^{n} p_i^{\frac{R}{2\alpha-\beta}}\right)^{\frac{2\alpha-\beta}{R}}\right], \alpha \geq 1,\ 0 < \beta \leq 1, \quad R(>0) \neq 1, 0 < R+\beta \neq 2\alpha. \tag{1.3}$$

(1.3) was called as the generalized R-norm information measure of type α and degree β and it reduces to (1.2) when $\alpha = 1$ and further reduces to (1.1) when $\beta = 1$.

Belis and Guiasu [8] considered qualitative aspect of events in an experiment and attached a utility distribution $U = \{(u_1, u_2, \ldots, u_n), u_i > 0\}$ with the probability distribution P such that u_i is the utility of an event having probability p_i. Consequently, the following 'qualitative-quantitative' measure was defined and characterized:

$$H(P;U) = H(p_1, p_2, \ldots, p_n; u_1, u_2, \ldots, u_n).$$

$$=-\sum_{i=1}^{n} u_i p_i \log p_i\ ,\ u_i > 0,\quad 0 < p_i \le 1,\quad \sum_{i=1}^{n} p_i = 1. \tag{1.4}$$

Later on the measure (1.4) was called 'useful' information by Longo [9] of the experiment.

To overcome the limitations of the measure (1.4), Bhaker and Hooda [10] introduced and characterized the following measure of 'useful' information:

$$H(P;U) = -\frac{\sum_{i=1}^{n} u_i p_i \log p_i}{\sum_{i=1}^{n} u_i p_i}. \quad . \tag{1.5}$$

Hooda et al. [11] also characterized the following 'useful' R-norm information measure:

$$H_R(P;U) = \frac{R}{R-1}\left[1-\left(\frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{1}{R}}\right],\ R(>0) \ne 1, \tag{1.6}$$

which reduces to Boekee and Lubbe [1] when utilities are ignored.

As the parametric generalized information measures have more potentiality and flexibility for applications point of view, so it is worthwhile to consider a two parametric generalized 'useful' R-norm information measure as given below:

$$H_R^{\alpha,\beta}(P;U) = \frac{R}{R+\beta-2\alpha}\left[1-\left(\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{2\alpha-\beta}{R}}\right],\ R(>0) \ne 1, \alpha \ge 1, 0 < \beta \le 1, 0 < R+\beta \ne 2\alpha, \tag{1.7}$$

where U is the utility distribution corresponding to probability distribution P. We may call (1.7) as the generalized 'useful' R-norm information measure of type α and degree β.

In the present paper the 'useful' R-norm information measure type α and degree β given by (1.7) is characterized in section 2. In section 3 the properties of 'useful' R-norm information measure of type α and degree β are studied. In section 4 the new measure is applied in studying the lower and upper bounds of a generalized 'useful' R-norm mean codeword length and in the end conclusion is also given as section 5.

2 Characterization of 'Useful' R-Norm Information Measure of Type α and Degree β

Let $S_n = \Delta_n \times \Delta_n^* \to R^+$, $n = 2,3,\ldots\ldots\ldots$ and G_n be a sequence of functions of $p_i{}'s$ and $u_i{}'s$, $i = 1,2,\ldots\ldots\ldots,n$, defined over S_n satisfying the following axioms:

Axiom 2.1 $G_n(P:U) = a_1 + a_2 \sum_{i=1}^{n} h(p_i, u_i)$, where a_1 and a_2 are non zero constants ration numbers and

$$p,u \in J = \{(0,1)\times(0,\infty)\} \cup \{(0,y);0 \le y \le 1\} \cup \{(\infty, y') : 0 \le y' \le \infty\}.$$

Axiom 2.2 For $P \in \Delta_n, U \in \Delta_n^*, P' \in \Delta_m,$ and $U' \in \Delta_m^*$, G_{mn} satisfies the following property:

$$G_{mn}(PP' : UU') = G_n(P:U) + G_m(P':U') - \frac{1}{a_1} G_n(P:U) G_m(P':U').$$

Axiom 2.3 $h(p,u)$ is a continuous function of its arguments p and u.

Axiom 2.4 Let all $p_i{}'s$ are equiprobable and $u_i{}'s$ are equal, then

$$G_n\left(\frac{1}{n},\ldots,\frac{1}{n};u,\ldots,u\right) = \frac{R}{R+\beta-2\alpha}\left[1-n^{\frac{-\beta+2\alpha-R}{R}}\right],$$

where $n = 2,3,\ldots,$ and $\alpha \ge 1$ $0 < \beta \le 1, R(>0) \ne 1, 0 < R+\beta \ne 2\alpha.$

Firstly, these lemmas are proved to facilitate the proof of the main theorem.

Lemma 2.1 Using axioms 2.1 and 2.2, we get the functional equation which is given below:

$$\sum_{i=1}^{n}\sum_{j=1}^{m} h\left(p_i p_j', u_i u_j'\right) = \left(\frac{-a_2}{a_1}\right) \sum_{i=1}^{n} h(p_i, u_i) \sum_{j=1}^{m} h\left(p_j', u_j'\right), \tag{2.1}$$

where $(p_i, u_i), (p_j', u_j') \in J$ for $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,m.$

Lemma 2.2 The continuous solution that satisfies (2.1) is the same as that of the functional equation:

$$h(pp', uu') = \left(\frac{-a_2}{a_1}\right) h(p,u)\, h(p',u'), \tag{2.2}$$

where a_1 and a_2 are arbitrary constants.

Proof: This Lemma can be proved on following the lines of Hooda et al. [11].

Next we obtain the general continuous solution of (2.2).

Lemma 2.3 One of the general continuous solutions of equation (2.2) is given by

$$h(p,u)=\left(\frac{-a_1}{a_2}\right)\left(\frac{p^{\mu}u^{\nu}}{pu}\right)^{1/\mu}, \text{ where } \mu\neq 0,\nu\neq 0 \tag{2.3}$$

and

$$h(p,u)=0. \tag{2.4}$$

Proof: Taking $g(p,u)=\left(\frac{-a_2}{a_1}\right)h(p,u)$ in (2.2), we have

$$g(pp',uu')=g(p,u)\,g(p',u'). \tag{2.5}$$

The most general continuous solution of (2.5) (refer to Aczel [12]) is given by

$$g(p,u)=\left(\frac{p^{\mu}u^{\nu}}{pu}\right)^{1/\mu}, \text{where } \mu\neq 0 \text{ and } \nu\neq 0. \tag{2.6}$$

and

$$g(p,u)=0. \tag{2.7}$$

On substituting $g(p,u)=\left(\frac{-a_2}{a_1}\right)h(p,u)$ in (2.6) and (2.7) we get (2.3) and (2.4) respectively. This proves the lemma 2.3 for all rational $p\in\,]0,1[$ and $u>0$, However, by continuity it holds for all real $p\in\,]0,1[$ and $u>0$.

Theorem 2.1 Axioms 2.1 to 2.4 together with Lemmas determine the measure (1.7)

Proof: Substituting the solution (2.3) in axiom 2.1, we have

$$G_n(P;U)=a_1\left[1-\sum_{i=1}^{n}\left(\frac{p_i^{\mu}u_i^{\nu}}{p_iu_i}\right)^{1/\mu}\right], \qquad \mu\nu\neq 0. \tag{2.8}$$

Taking $p_i = \frac{1}{n}$ and $u_i = u$ for each i in (2.8), we get

$$G_n\left(\frac{1}{n},\ldots,\frac{1}{n},u,\ldots,u\right)=a_1\left(1-n^{\frac{1-\mu}{\mu}}u^{\frac{\nu-1}{\mu}}\right),\qquad n=2,3,\ldots, \tag{2.9}$$

Axiom 2.4 together with (2.9) gives

$$a_1\left(1-n^{\frac{1-\mu}{\mu}}u^{\frac{\nu-1}{\mu}}\right)=\frac{R}{R+\beta-2\alpha}\left[1-n^{\frac{-\beta+2\alpha-R}{R}}\right].$$

It implies

$$a_1=\frac{R}{R+\beta-2},\quad \mu=\frac{R}{2\alpha-\beta},\quad \nu=1.$$

Putting these values in (2.8), we have

$$G_n(P;U)=\frac{R}{R+\beta-2\alpha}\left[1-\left(\frac{\sum_{i=1}^{n}u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^{n}u_i p_i}\right)^{2\alpha-\beta/R}\right]$$
$$=H_R^{\alpha,\beta}(P;U).$$

Hence this completes the proof of theorem 2.1.

3 Main Properties of $H_R^{\alpha,\beta}(P;U)$

The following properties are satisfied by the generalized ‘useful’ R-norm information measure $H_R^{\alpha,\beta}(P;U)$

Property 3.1: $H_R^{\alpha,\beta}(P;U)$ is a symmetric function of their arguments, if permutation of p_i^s and u_i^s are taken together, i.e.

$$H_R^{\alpha,\beta}(p_1,p_2,\ldots,p_{n-1},p_n;u_1,u_2,\ldots u_{n-1},u_n)=H_R^{\alpha,\beta}(p_n,p_1,p_2,\ldots,p_{n-1};u_n,u_1,u_2,\ldots,u_{n-1})..$$

Property 3.2: $H_R^{\alpha,\beta}\left(\frac{1}{8},\frac{1}{8};1,1\right)=1$, in case $\alpha=\beta=1$ and $R=2$.

Proof: $H_R^{\alpha,\beta}(P;U)=\dfrac{R}{R+\beta-2\alpha}\left[1-\left(\dfrac{\sum_{i=1}^{n}u_i p_i^{\frac{2\alpha-\beta}{R}}}{\sum_{i=1}^{n}u_i p_i}\right)^{\frac{R}{2\alpha-\beta}}\right]$

For $i=1,2,$ we have

$$H_R^{\alpha,\beta}(P;U)=\frac{R}{R+\beta-2\alpha}\left[1-\left(\frac{u_1 p_1^{\frac{R}{2\alpha-\beta}}}{u_1 p_1}+\frac{u_2 p_2^{\frac{R}{2\alpha-\beta}}}{u_2 p_2}\right)^{\frac{2\alpha-\beta}{R}}\right].$$

Setting

$p_1=\dfrac{1}{8}$, $p_2=\dfrac{1}{8}$ $,u_1=1$, $u_2=1$ $,\alpha=1,\beta=1$ and $R=2.$, we get

$$H_2^{1,1}\left(\frac{1}{8},\frac{1}{8};1,1\right)=2\left[1-\left\{\frac{\left(\frac{1}{8}\right)^2}{\frac{1}{8}}+\frac{\left(\frac{1}{8}\right)^2}{\frac{1}{8}}\right\}^{\frac{1}{2}}\right]$$

$$=2\left[1-\left(\frac{1}{4}\right)^{\frac{1}{2}}\right]=1.$$

Property 3.3: Addition of one event having probability of occurrence as zero or utility as zero has no effect on 'useful' information, i.e.

$$H_R^{\alpha,\beta}(p_1,p_2,\ldots,p_n,0;u_1,u_2,\ldots,u_{n+1})=H_R^{\alpha,\beta}(p_1,p_2,\ldots,p_n;u_1,u_2,\ldots,u_n)$$
$$=H_R^{\alpha,\beta}(p_1,p_2,\ldots,p_{n+1};u_1,u_2,\ldots,u_n,0).$$

Proof: Let us consider

$$H_R^{\alpha,\beta}(p_1,p_2,\ldots,p_n,0;u_1,u_2,\ldots,u_{n+1})$$
$$=\frac{R}{R+\beta-2\alpha}\left[1-\left\{\frac{u_1 p_1^{\frac{R}{2\alpha-\beta}}}{u_1 p_1}+\frac{u_2 p_2^{\frac{R}{2\alpha-\beta}}}{u_2 p_2}+\ldots+\frac{u_n p_n^{\frac{R}{2\alpha-\beta}}}{u_n p_n}+\ldots+0\right\}^{\frac{2\alpha-\beta}{R}}\right]$$
$$=H_R^{\alpha,\beta}(P;U).$$

Property 3.4: $H_R^{\alpha,\beta}(P;U)$ satisfies the non additivity of the following form:

$$H_R^{\alpha,\beta}(P*Q;U*V)=H_R^{\alpha,\beta}(P;U)+H_R^{\alpha,\beta}(Q;V)-\frac{R+\beta-2\alpha}{R}H_R^{\alpha,\beta}(P;U)H_R^{\alpha,\beta}(Q;V),$$

where $P*Q=(p_1q_1,\ldots,p_1q_m;p_2q_1,\ldots,p_2q_m,p_nq_1,\ldots,p_nq_m)$,

$$U*V=(u_1v_1,\ldots,u_1v_m,u_2v_1,\ldots,u_2v_m,u_nv_1,\ldots,u_nv_m).$$

Proof: R.H.S. $=H_R^{\alpha,\beta}(P;U)+H_R^{\alpha,\beta}(Q;V)-\frac{R+\beta-2\alpha}{R}H_R^{\alpha,\beta}(P;U)H_R^{\alpha,\beta}(Q;V)$

$$=\frac{R}{R+\beta-2\alpha}\left[1-\left\{\frac{\sum_{i=1}^{n}u_iv_j(p_iq_j)^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^{n}u_iv_jp_iq_j}\right\}^{\frac{2\alpha-\beta}{R}}\right]$$

$$=H_R^{\alpha,\beta}(P:Q;U:V)$$
$$=\text{L.H.S.}$$

Property 3.5: Let A_i,A_j be two events having probabilities p_i,p_j and utilities u_i,u_j respectively, then the utility u of compound event $A_i\cap A_j$ *is* defined

$$u(A_i\cap A_j)=\frac{u_ip_i+u_jp_j}{p_i+p_j}. \tag{3.1}$$

Theorem 3.1: under the composition law (3.1) following holds:

$${}_{n+1}H_R^{\beta}(p_1,p_2,\ldots,p_{n+1},p',p'';u_1,u_2,\ldots,u_{n+1},u',u'')$$

$$={}_nH_R^{\alpha,\beta}(p_1,p_2,\ldots,p_{n+1};u_1,u_2,\ldots,u_{n+1})+(p'+p'')H_R^{\alpha,\beta}\left(\frac{p'}{p'+p''},\frac{p''}{p'+p''};u',u''\right).$$

Proof: ${}_{n+1}H_R^{\alpha,\beta}(p_1,p_2,\ldots,p_{n+1},p',p'';u_1,u_2,\ldots,u_{n+1}u',u'')$

$$={}_nH_R^{\alpha,\beta}(p_1,p_2,\ldots,p_{n+1};u_1,u_2,\ldots,u_{n+1})+\frac{R}{R+\beta-2\alpha}\left[1-\left\{\frac{u'p'^{\frac{R}{2\alpha-\beta}}}{u'p'}+\frac{u''p''^{\frac{R}{2\alpha-\beta}}}{u''p''}\right\}^{\frac{2\alpha-\beta}{R}}\right]$$

$$=_n H_R^{\alpha,\beta} + \frac{R}{R+\beta-2\alpha}\left[\left(p'+p''\right)-\left\{\frac{u'\left(\frac{p'}{p'+p''}\right)^{\frac{R}{2\alpha-\beta}}}{u'\frac{p'}{p'+p''}}+\frac{u''\left(\frac{p''}{p'+p''}\right)^{\frac{R}{2\alpha-\beta}}}{u''\frac{p''}{p'+p''}}\right\}^{\frac{2\alpha-\beta}{R}}\left(p'+p''\right)\right]$$

$$=_n H_R^{\alpha,\beta} + \left(p'+p''\right) H_R^{\alpha,\beta}\left[\frac{p'}{p'+p''},\frac{p''}{p'+p''};u',u''\right].$$

This completes the proof of theorem 3.1.

4 Application in Source Coding

Here we introduce a new generalized ‘useful’ mean code word length as given below:

$$L_R^{\alpha,\beta}\left(P;U\right)=\frac{R}{R+\beta-2\alpha}\left[1-\frac{\sum_{i=1}^{n}u_i p_i D^{-l_i\left(\frac{R+\beta-2\alpha}{R}\right)}}{\sum_{i=1}^{n}u_i p_i}\right], \tag{4.1}$$

In case $\alpha=1$ and $\beta=1$, (4.1) reduces to average code word length that was given by Hooda et al. [11] and again if utilities are ignored, i.e. i=1 for all $i=1,2,\dots,n$, it reduces to

$$L_R\left(P\right)=\frac{R}{R-1}\left[1-\sum_{i=1}^{n}p_i D^{-l_i\left(\frac{R-1}{R}\right)}\right], \tag{4.2}$$

which is average codeword length due to Boekee and Lubbe [1]. Thus (4.1) is a valid non-additive ‘useful’ mean codeword length.

Next we study the lower and upper bounds of $L_R^{\alpha,\beta}\left(P;U\right)$) in terms of ‘useful’ R-norm information measure of type α and degree β $H_R^{\alpha,\beta}\left(P;U\right)$ given by (1.7).

Theorem 4.1: If $l_i, i=1,2,\dots,n$ is length of codewords $x_i's$, then

$$H_R^{\alpha,\beta}\left(P;U\right)\le L_R^{\alpha,\beta}\left(P;U\right), R(\ne 1)>0, \alpha\ge 1, 0<\beta\le 1, \tag{4.3}$$

under the condition

$$\sum_{i=1}^{n}u_i D^{-l_i}\le\sum_{i=1}^{n}u_i p_i, \tag{4.4}$$

where (4.4) is the generalization of Kraft's inequality [13].

Proof: By Holder's inequality we have

$$\left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} y_i^q\right)^{\frac{1}{q}} \le \sum_{i=1}^{n} x_i y_i\,, \tag{4.5}$$

where $x_i, y_i \ge 0$ for each i and $\frac{1}{p}+\frac{1}{q}=1.$

Setting $x_i = \left(\frac{u_i p_i}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{R}{R+\beta-2\alpha}} D^{-l_i}$, $y_i = \left(\frac{u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{2\alpha-\beta}{2\alpha-\beta-R}}$

$p = \frac{R+\beta-2\alpha}{R}$ and $q = \frac{2\alpha-\beta-R}{2\alpha-\beta}$ and putting in (4.5), we have

$$\left(\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R+\beta-2\alpha}{R}\right)}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{R}{R+\beta-2\alpha}} \left(\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{2\alpha-\beta}{2\alpha-\beta-R}} \le \frac{\sum_{i=1}^{n} u_i D^{-l_i}}{\sum_{i=1}^{n} u_i p_i} \le 1.$$

It implies

$$\left(\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{2\alpha-\beta}{2\alpha-\beta-R}} \le \left(\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R+\beta-2\alpha}{R}\right)}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{R}{2\alpha-\beta-R}}. \tag{4.6}$$

Case 1 When $R+\beta<2\alpha$, raising Power $\frac{2\alpha-\beta-R}{R}>0$ both sides of (4.6), we have

$$\left(\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{2\alpha-\beta}{R}} \le \frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R+\beta-2\alpha}{R}\right)}}{\sum_{i=1}^{n} u_i p_i}.$$

Subtracting both sides from 1, we get

$$1-\left(\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{2\alpha-\beta}{R}} \geq 1-\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R+\beta-2\alpha}{R}\right)}}{\sum_{i=1}^{n} u_i p_i}. \tag{4.7}$$

Multiplying (4.7) by $\frac{R}{R+\beta-2\alpha}<0$ both sides, we have

$$\frac{R}{R+\beta-2\alpha}\left[1-\left(\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{2\alpha-\beta}{R}}\right] \leq \frac{R}{R+\beta-2\alpha}\left[1-\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R+\beta-2\alpha}{R}\right)}}{\sum_{i=1}^{n} u_i p_i}\right]$$

$$H_R^{\alpha,\beta}(P;U) \leq L_R^{\alpha,\beta}(P;U). \tag{4.8}$$

Case 2 When $R+\beta>2\alpha$, Raising power $\frac{2\alpha-\beta-R}{R}<0$ both sides of (4.6), we get

$$\left(\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{2\alpha-\beta}{R}} \geq \frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R+\beta-2\alpha}{R}\right)}}{\sum_{i=1}^{n} u_i p_i}.$$

Subtracting both sides from 1, we get

$$1-\left(\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R}{2-\beta}}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{2\alpha-\beta}{R}} \leq 1-\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R+\beta-2}{R}\right)}}{\sum_{i=1}^{n} u_i p_i}. \tag{4.9}$$

Multiplying (4.9) by $\frac{R}{R+\beta-2\alpha}>0$ both sides, we get

$$\frac{R}{R+\beta-2\alpha}\left[1-\left(\frac{\sum_{i=1}^{n}u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^{n}u_i p_i}\right)^{\frac{2\alpha-\beta}{R}}\right]\le\frac{R}{R+\beta-2\alpha}\left[1-\frac{\sum_{i=1}^{n}u_i p_i D^{-l_i\left(\frac{R+\beta-2\alpha}{R}\right)}}{\sum_{i=1}^{n}u_i p_i}\right]$$

$$H_R^{\alpha,\beta}(P;U)\le L_R^{\alpha,\beta}(P;U). \tag{4.10}$$

Thus theorem 4.1 is proved in both cases.

In (4.3) equality holds if and only if

$$D^{-l_i}=\frac{p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^{n}u_i p_i^{\frac{R}{2\alpha-\beta}}\Big/\sum_{i=1}^{n}u_i p_i},\ R>0(\ne 1)$$

or

$$l_i=-\log_D p_i^{\frac{R}{2\alpha-\beta}}+\log_D\left[\frac{\sum_{i=1}^{n}u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^{n}u_i p_i}\right]$$

$$\log_D p_i^{-\frac{R}{2\alpha-\beta}}\left[\frac{\sum_{i=1}^{n}u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^{n}u_i p_i}\right]\le l_i<\log_D p_i^{-\frac{R}{2\alpha-\beta}}\left[\frac{\sum_{i=1}^{n}u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^{n}u_i p_i}\right]+1. \tag{4.11}$$

It implies

$$p_i^{-\frac{R}{2\alpha-\beta}}\left[\frac{\sum_{i=1}^{n}u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^{n}u_i p_i}\right]\le D^{l_i}<Dp_i^{-\frac{R}{2\alpha-\beta}}\left[\frac{\sum_{i=1}^{n}u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^{n}u_i p_i}\right]. \tag{4.12}$$

In the next theorem, we obtain an upper bound on $L_R^{\alpha,\beta}(P;U)$ in terms of $H_R^{\alpha,\beta}(P;U)$.

Theorem 4.2: Let $l_1, l_2, \ldots, l_n$ be the codeword lengths satisfying (4.12), then following inequality holds:

$$L_R^{\alpha,\beta}(P;U) \le D^{\frac{2\alpha-\beta-R}{R}} H_R^{\alpha,\beta}(P;U) + \frac{R}{R+\beta-2\alpha}\left(1-D^{\frac{2\alpha-\beta-R}{R}}\right). \tag{4.13}$$

Proof: From the R.H.S. of (4.12), we have

$$D^{l_i} < D p_i^{-\frac{R}{2\alpha-\beta}} \left[\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^{n} u_i p_i}\right]. \tag{4.14}$$

Here two cases arise:

Case 1 When $0 < R+\beta < 2\alpha$, raising both sides of (4.14) to the power $\frac{2\alpha-\beta-R}{R} > 0$, we get

$$D^{-l_i\left(\frac{R+\beta-2\alpha}{R}\right)} < D^{\frac{2\alpha-\beta-R}{R}} p_i^{\frac{R+\beta-2\alpha}{2\alpha-\beta}} \left[\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^{n} u_i p_i}\right]^{\frac{2\alpha-\beta-R}{R}}. \tag{4.15}$$

Multiplying both sides of (4.15) by $\frac{u_i p_i}{\sum_{i=1}^{n} u_i p_i}$ and summing over i, we have

$$\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R+\beta-2\alpha}{R}\right)}}{\sum_{i=1}^{n} u_i p_i} < D^{\frac{2\alpha-\beta-R}{R}} \left[\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^{n} u_i p_i}\right]^{\frac{2\alpha-\beta-R}{R}} \left[\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^{n} u_i p_i}\right]$$

or

$$\frac{\sum_{i=1}^{n} u_i p_i D^{-l_i\left(\frac{R+\beta-2\alpha}{R}\right)}}{\sum_{i=1}^{n} u_i p_i} < D^{\frac{2\alpha-\beta-R}{R}} \left[\frac{\sum_{i=1}^{n} u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^{n} u_i p_i}\right]^{\frac{2\alpha-\beta}{R}}.$$

Subtracting both sides from 1 and multiplying by $\dfrac{R}{R+\beta-2\alpha}<0$, we have

$$L_R^{\alpha,\beta}\left(P;U\right)<D^{\frac{2\alpha-\beta-R}{R}}H_R^{\alpha,\beta}\left(P;U\right)+\frac{R}{R+\beta-2\alpha}\left[1-D^{\frac{2\alpha-\beta-R}{R}}\right]. \tag{4.16}$$

Similarly, we can prove that (4.16) holds when $R+\beta>2\alpha$. Hence theorem 4.2 is proved

Thus (4.3) and (4.13) together give

$$H_R^{\alpha,\beta}\left(P:U\right)\leq L_R^{\alpha,\beta}\left(P;U\right)<D^{\frac{2\alpha-\beta-R}{R}}H_R^{\alpha,\beta}\left(P;U\right)+\frac{R}{R+\beta-2\alpha}\left(1-D^{\frac{2\alpha-\beta-R}{R}}\right).$$

5 Conclusion

In this paper we have defined a new generalized 'useful' R-norm information measure analogous to Hooda and Sharma's [7] R-norm information measure and characterized axiomatically. Some important properties have also been studied.

The new generalized measure has been applied in obtaining lower and upper bounds of the generalized 'useful' mean code word lengths. This can be generalized parametrically and applied in source coding. The measure can be further generalized parametrically and can be applied in source coding.

Competing Interests

Authors have declared that no competing interests exist.

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