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Integration of First-order Modeled Differential Equations Using a Quarter-step Method

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Authors' contributions

This work was carried out in collaboration between all authors. Author DY derived the quarter-step method using Laguerre polynomial. Author JNA analyzed the basis properties of the quarter-step method derived while author JS implemented the quarter-step method on sampled first order problems with the aid of MATLAB programming language. All authors read and approved the final manuscript.

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ABSTRACT

In this paper, we present the derivation and implementation of a quarter-step method for the integration of first-order modeled differential equations. The quarter-step method was developed using Laguerre polynomial of degree six as our basis function via interpolation and collocation techniques. We went further to apply the quarter-step method developed on some modeled first order differential equations. The paper also analyzed the basic properties of the method derived. From the results obtained, it is obvious that the method is computationally reliable.

Keywords: First-order; integration; hybrid; laguerre polynomial; quarter-step; model.

2010 AMS subject classification: 65L05, 65L06, 65D30.

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1. INTRODUCTION

This paper presents a quarter-step method for the integration of modeled first order problems of the form,

$$y' = f(x,y), y(a) = \eta, f : R \times R \to R \qquad (1)$$

where $f(x,y)$ is assumed to satisfy Lipschitz condition which guarantees the existence and uniqueness of the solutions of (1).

Definition 1.1 [1]

Laguerre polynomial $y_n(x)$ is defined as,

$$\sum_{n=0}^{\infty} y_n(x) = \sum_{n=0}^{\infty} \frac{e^x}{n!} \frac{d^n}{dx^n}\left(x^n e^{-x}\right) \qquad (2)$$

In particular,

$$\left.\begin{array}{l} y_0(x) = 1 \\ y_1(x) = x - 1 \\ y_2(x) = x^2 - 4x + 2 \\ \quad . \\ \quad . \\ \quad . \end{array}\right\} \qquad (3)$$

It is important to note that the Laguerre polynomial $y_n(x)$ are orthogonal with respect to the weight function $w(x) = e^{-x}$ on $[0,\infty)$.

Many scholars used different basis functions for the solution of problems of the form (1). For instance, the authors in [2] and [3] used basis functions which are the combination of power series and exponential functions to develop block integrators for the solution of (1). The authors in [4] and [5] also used Chebyshev and Legendre polynomials as basis functions respectively to develop hybrid methods for the solution of (1).

In this paper, we shall employ Laguerre polynomial of degree 6 as a basis function in developing the quarter-step method for the solution of (1).

2. METHODOLOGY: DERIVATION OF THE QUARTER-STEP METHOD

We shall derive a quarter-step method of the form,

$$A^{(0)}\mathbf{Y}_m = E\mathbf{y}_n + hd\mathbf{f}(\mathbf{y}_n) + hb\mathbf{F}(\mathbf{Y}_m) \qquad (4)$$

using Laguerre polynomial of degree 6 as our basis function. This is given by,

$$y_6(x) = 5040 - 1512x + 12600x^2 - 4200x^3 + 630x^4 - 42x^5 + x^6 \qquad (5)$$

We interpolate (5) at point $x_{n+s}, s = 0$ and collocate its first derivative at points $x_{n+r}, r = 0\left(\frac{1}{20}\right)\frac{1}{4}$, where s and r are the numbers of interpolation and collocation points respectively, this leads to the system of equations of the form,

$$XA = U \qquad (6)$$

where

$$A = [a_0 \; a_1 \; a_2 \; a_3 \; a_4 \; a_5 \; a_6]^T \qquad U = \left[y_n \;\; f_n \;\; f_{n+\frac{1}{20}} \;\; f_{n+\frac{1}{10}} \;\; f_{n+\frac{3}{20}} \;\; f_{n+\frac{1}{5}} \;\; f_{n+\frac{1}{4}} \right]^T$$

and

$$X = \begin{bmatrix} 5040 & -1512x_n & 12600x_n^2 & -4200x_n^3 & 630x_n^4 & -42x_n^5 & x^6 \\ 0 & -1512 & 25200x_n & -12600x_n^2 & 2520x_n^3 & -210x_n^4 & 6x_n^5 \\ 0 & -1512 & 25200x_{n+\frac{1}{20}} & -12600x^2_{n+\frac{1}{20}} & 2520x^3_{n+\frac{1}{20}} & -210x^4_{n+\frac{1}{16}} & 6x^5_{n+\frac{1}{20}} \\ 0 & -1512 & 25200x_{n+\frac{1}{10}} & -12600x^2_{n+\frac{1}{10}} & 2520x^3_{n+\frac{1}{10}} & -210x^4_{n+\frac{1}{10}} & 6x^5_{n+\frac{1}{10}} \\ 0 & -1512 & 25200x_{n+\frac{3}{20}} & -12600x^2_{n+\frac{3}{20}} & 2520x^3_{n+\frac{3}{20}} & -210x^4_{n+\frac{3}{16}} & 6x^5_{n+\frac{3}{20}} \\ 0 & -1512 & 25200x_{n+\frac{1}{5}} & -12600x^2_{n+\frac{1}{5}} & 2520x^3_{n+\frac{1}{5}} & -210x^4_{n+\frac{1}{5}} & 6x^5_{n+\frac{1}{5}} \\ 0 & -1512 & 25200x_{n+\frac{1}{4}} & -12600x^2_{n+\frac{1}{4}} & 2520x^3_{n+\frac{1}{4}} & -210x^4_{n+\frac{1}{4}} & 6x^5_{n+\frac{1}{4}} \end{bmatrix}$$

Solving (6) for $a_j\text{'s}, j=0(1)6$ and substituting back into the basis function gives a continuous linear multistep method of the form,

$$y(x)=\alpha_0(x)y_n+h\sum_{j=0}^{\frac{1}{4}}\beta_j(x)f_{n+j},\ j=0\left(\frac{1}{20}\right)\frac{1}{4} \tag{7}$$

where

$$\left.\begin{aligned}
&\alpha_0=1\\
&\beta_0=-\frac{1}{18}(80000t^6-72000t^5+25500t^4-4500t^3+411t^3-18t)\\
&\beta_{\frac{1}{20}}=\frac{50}{90}(4000t^6-3360t^5+1065t^4-154t^3+9t^2)\\
&\beta_{\frac{1}{10}}=-\frac{50}{9}(8000t^6-6240t^5+1770t^4-214t^3+9t^2)\\
&\beta_{\frac{3}{20}}=\frac{100}{9}(4000t^6-2880t^5+735t^4-78t^3+3t^2)\\
&\beta_{\frac{1}{5}}=-\frac{25}{18}(16000t^6-10560t^5+2460t^4-244t^3+9t^2)\\
&\beta_{\frac{1}{4}}=\frac{2}{9}(20000t^6-12000t^5+2625t^4-250t^3+9t^2)
\end{aligned}\right\} \tag{8}$$

$t=\dfrac{x-x_n}{h}$, $\alpha(t)$ *and* $\beta(t)$ are continuous functions. Evaluating (7) at $t=\frac{1}{20}\left(\frac{1}{20}\right)\frac{1}{4}$, gives a discrete block method of the form (4), where

$$\mathbf{Y}_m=\begin{bmatrix} y_{n+\frac{1}{20}} & y_{n+\frac{1}{10}} & y_{n+\frac{3}{20}} & y_{n+\frac{1}{5}} & y_{n+\frac{1}{4}}\end{bmatrix}^T \quad \mathbf{y}_n=\begin{bmatrix} y_{n-\frac{1}{20}} & y_{n-\frac{3}{20}} & y_{n-\frac{1}{10}} & y_{n-\frac{1}{20}} & y_n\end{bmatrix}^T$$

$$\mathbf{F}(\mathbf{Y}_m)=\begin{bmatrix} f_{n+\frac{1}{20}} & f_{n+\frac{1}{10}} & f_{n+\frac{3}{20}} & f_{n+\frac{1}{5}} & f_{n+\frac{1}{4}}\end{bmatrix}^T \quad \mathbf{f}(\mathbf{y}_n)=\begin{bmatrix} f_{n-\frac{1}{5}} & f_{n-\frac{3}{20}} & f_{n-\frac{1}{10}} & f_{n-\frac{1}{20}} & f_n\end{bmatrix}^T$$

$$A^{(0)}=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}\quad E=\begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix}\quad d=\begin{bmatrix}0&0&0&0&\frac{19}{1152}\\0&0&0&0&\frac{7}{450}\\0&0&0&0&\frac{51}{3200}\\0&0&0&0&\frac{7}{450}\\0&0&0&0&\frac{19}{1152}\end{bmatrix}$$

$$b=\begin{bmatrix} \frac{1427}{28800} & \frac{-133}{4800} & \frac{241}{14400} & \frac{-173}{28800} & \frac{3}{3200} \\ \frac{43}{600} & \frac{7}{900} & \frac{7}{900} & \frac{-1}{300} & \frac{1}{1800} \\ \frac{219}{3200} & \frac{57}{1600} & \frac{57}{1600} & \frac{-21}{3200} & \frac{3}{3200} \\ \frac{16}{225} & \frac{2}{75} & \frac{16}{225} & \frac{7}{450} & 0 \\ \frac{25}{384} & \frac{25}{576} & \frac{25}{576} & \frac{25}{384} & \frac{19}{1152} \end{bmatrix}$$

3. ANALYSIS OF BASIC PROPERTIES OF THE QUARTER-STEP METHOD

To justify the applicability and accuracy of the proposed method, we need to examine its basic properties which include order of accuracy, consistency, root condition, convergence, symmetry and region of absolute stability.

3.1 Order of Accuracy and Error Constant

The block method (4) is said to be uniform accurate order *p*, if *p* is the largest positive integer for which $\bar{c}_0=\bar{c}_1=\bar{c}_2=...=\bar{c}_p=0$ but $\bar{c}_{p+1}\neq 0,$ [6]. Thus, for our method,

$$\bar{c}_0=\bar{c}_1=\bar{c}_2=\bar{c}_3=\bar{c}_4=\bar{c}_5=\bar{c}_6=0,$$
$$\bar{c}_7=\begin{bmatrix}-1.1148\times10^{-11} & -7.6472\times10^{-12} & -1.0114\times10^{-11} & -6.6138\times10^{-12} & -1.0114\times10^{-11}\end{bmatrix}^T$$

Therefore, the quarter-step method is of uniform sixth order.

3.2 Root Condition and Zero Stability

Definition 3.1 [6]: The block method (4) is said to satisfy root condition, if the roots $z_{s,}\, s=1,2,...,k$ of the first characteristic polynomial $\rho(z)$ defined by $\rho(z)=\det(zA^{(0)}-E)$ satisfies $|z_s|\leq 1$ and every root satisfying $|z_s|=1$ have multiplicity not exceeding the order of the differential equation. The method (4) is said to be zero-stable if it satisfies the root condition.

$$\rho(z)=\left|z\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix}\right| \tag{9}$$

$$\rho(z)=z^4(z-1)=0\Rightarrow z_1=z_2=z_3=z_4=0,\; z_5=1.$$

Hence, the quarter-step method (4) is said to satisfy root condition.

Theorem 3.1 [6]: *The necessary and sufficient condition for the method given by (4) to be zero-stable is that it satisfies the root condition.*

3.3 Consistency

According to [7], consistency controls the magnitude of the local truncation error committed at each stage of the computation. The quarter-step method (4) is consistent since it has order $p=6>1$

3.4 Convergence

The quarter-step method (4) is convergent by consequence of Dahlquist theorem below.

Theorem 3.2 [8]: *The necessary and sufficient conditions that a continuous LMM be convergent are that it be consistent and zero-stable.*

3.5 Region of Absolute Stability

In the plotting the stability region, we shall adopt the boundary locus method. The stability polynomial of the quarter-step method is given by,

$$\bar{h}(w)=-h^5\left(\frac{1}{19200000}w^5+\frac{73}{2073600000}w^4\right)-h^4\left(\frac{15271}{3110400000}w^4-\frac{137}{28800000}w^5\right)$$
$$-h^3\left(\frac{3}{12800}w^5+\frac{103}{460800}w^4\right)+h^2\left(\frac{17}{2400}w^5-\frac{7469}{1036800}w^4\right)-h\left(\frac{1}{8}w^5+\frac{1}{8}w^4\right)+w^5-w^4 \qquad (10)$$

The stability region is shown in Fig. 1.

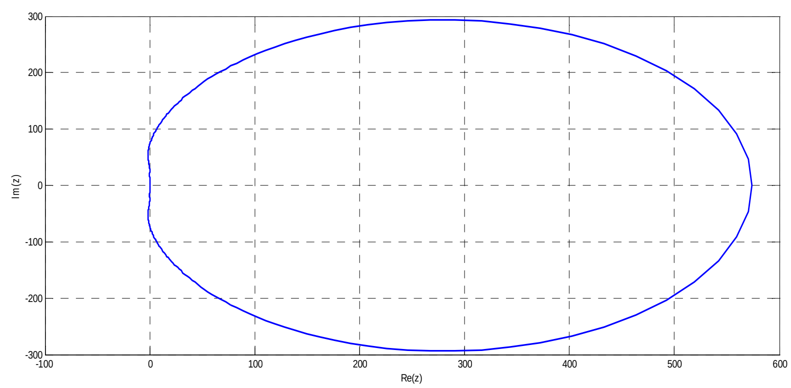

Fig. 1. Stability region of the quarter-step method

The RAS obtained in Fig. 1 is A-stable, since it contains the whole of the left-half complex plane, [6].

4. RESULTS

4.1 Numerical Experiments

We shall consider the following two linear real-life problems by modeling them into equations of the form (1). A nonlinear problem shall also be considered. We shall use the following notation in the tables below.

ERR= |Exact Solution – Computed Solution|
Eval *t* =Evaluation time per seconds
ESYA=Error in [9]
ESOJA=Error in [3]

Problem 4.1 *(Growth Model)*

A bacteria culture is known to grow at a rate proportional to the amount present. After one hour, 1000 strands of the bacteria are observed in the culture; and after four hours, 3000 strands. Find the number of strands of the bacteria present in the culture at time $t: 0 \le t \le 1$.

Let $N(t)$ denote the number of bacteria strands in the culture at time t, the initial value problem modeling this problem is given by,

$$\frac{dN}{dt} = 0.366N, \quad N(0) = 694 \tag{11}$$

The exact solution is give by,

$$N(t) = 694e^{0.366t} \tag{12}$$

Source: [10]

The authors in [9] solved this problem by applying a quarter-step method of order 5. We compare the result obtained using our method with theirs as shown in Table 4.1.

Problem 4.2 *(Electric Circuit Model)*

A 12V battery is connected to a series circuit in which the inductance is $\frac{1}{2}H$ and the resistance is 10Ω. Determine the current i *if* $i(0) = 0$ at time $t: 0 < t \le 0.1$.

If a circuit has in series an emf E volt, a resistor R Ohm and an inductor L Henries, then the current i in amperes at time t is given by,

$$L\left(\frac{di}{dt}\right) + Ri = E \tag{13}$$

Thus, the initial value problem modeling the problem is given by,

$$\frac{di}{dt} = -20i + 24, \; i(0) = 0 \tag{14}$$

with the exact solution,

$$i(t) = \left(\frac{6}{5}\right)\left(1 - e^{-20t}\right) \tag{15}$$

Source: [1]

The authors in [9] solved this problem by applying a quarter-step method of order 5. We compare the result obtained using our method with theirs as shown in Table 4.2.

Problem 4.3 *(Non-Linear Problem)*

Consider the nonlinear problem below,

$$\frac{dy}{dx} = -10(y-1)^2, \; y(0) = 2 \tag{16}$$

with the exact solution

$$y(x) = 1 + \frac{1}{1+10x} \tag{17}$$

Source: [3]

The authors in [3] solved this problem by applying a numerical method of order 7. We compare the result obtained using our method with theirs as shown in Table 4.3.

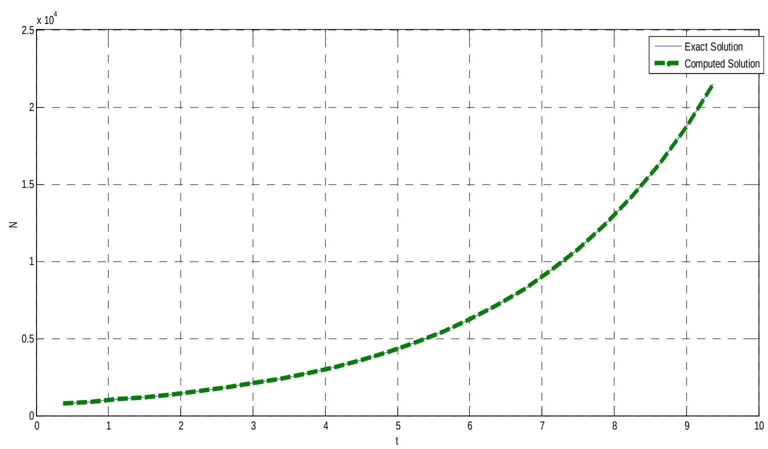

Fig. 2. Graphical result for problem 4.1 (Growth model)

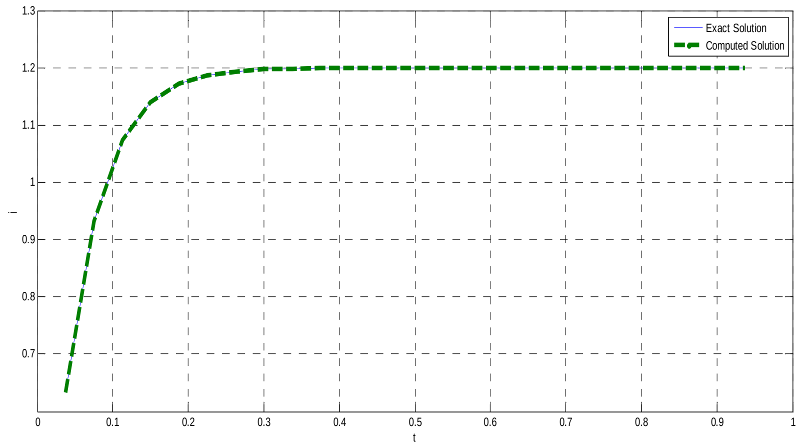

Fig. 3. Graphical result for problem 4.2 (Electric circuit model)

Table 4.1. Showing the result for problem 4.1

t	Exact solution	Computed solution	ERR	ESYA	Eval t
0.10	719.8709504841319800	719.8709504841319800	0.000000e+000	0.000000e+000	0.0764
0.20	746.7063189494632500	746.7063189494632500	0.000000e+000	0.000000e+000	0.0785
0.30	774.5420569951836600	774.5420569951836600	0.000000e+000	0.000000e+000	0.0807
0.40	803.4154564251550700	803.4154564251550700	0.000000e+000	0.000000e+000	0.0827
0.50	833.3651992080965600	833.3651992080965600	0.000000e+000	0.000000e+000	0.0848
0.60	864.4314093001880800	864.4314093001880800	0.000000e+000	2.273737e-013	0.0870
0.70	896.6557063995159100	896.6557063995159100	0.000000e+000	2.273737e-013	0.0891
0.80	930.0812617043808400	930.0812617043808400	0.000000e+000	3.410605e-013	0.0911
0.90	964.7528557501631200	964.7528557501631200	0.000000e+000	2.273737e-013	0.0931
1.00	1000.7169384022342000	1000.7169384022342000	0.000000e+000	3.410605e-013	0.0953

Table 4.2. Showing the result for problem 4.2

i	Exact solution	Computed solution	ERR	ESYA	Eval t
0.01	0.2175230963064218	0.2175230963064218	0.000000e+000	6.364631e-013	0.0226
0.02	0.3956159447572328	0.3956159447572328	0.000000e+000	1.042055e-012	0.0246
0.03	0.5414260366871682	0.5414260366871682	0.000000e+000	1.279643e-012	0.0266
0.04	0.6608052430593341	0.6608052430593341	0.000000e+000	1.397105e-012	0.0286
0.05	0.7585446705942693	0.7585446705942693	0.000000e+000	1.429967e-012	0.0306
0.06	0.8385669457053576	0.8385669457053576	0.000000e+000	1.404876e-012	0.0327
0.07	0.9040836432700723	0.9040836432700723	0.000000e+000	1.341927e-012	0.0348
0.08	0.9577241784064137	0.9577241784064137	0.000000e+000	1.255662e-012	0.0368
0.09	1.0016413341340962	1.0016413341340962	0.000000e+000	1.156408e-012	0.0388
0.10	1.0375976601160648	1.0375976601160648	0.000000e+000	1.052047e-012	0.0408

Table 4.3. Showing the result for problem 4.3

x	Exact solution	Computed solution	ERR	ESOJA	Eval t
0.01	1.9090909090909092	1.9090886423023135	2.266789e-006	1.07e-03	0.0139
0.02	1.8333333333333335	1.8333312639246935	2.069409e-006	2.38e-03	0.0151
0.03	1.7692307692307692	1.7692184876881401	1.228154e-005	2.21e-03	0.0162
0.04	1.7142857142857144	1.7141948078088534	9.090648e-005	5.36e-03	0.0173
0.05	1.6666666666666665	1.6661926448958333	4.740218e-004	7.53e-03	0.0185
0.06	1.6250000000000000	1.6245832366132822	4.167634e-004	9.00e-03	0.0191
0.07	1.5882352941176470	1.5878661146067676	3.691795e-004	9.98e-03	0.0192
0.08	1.5555555555555556	1.5552259169035829	3.296387e-004	1.06e-02	0.0194
0.09	1.5263157894736841	1.5260174514784275	2.983380e-004	1.10e-02	0.0195
0.10	1.5000000000000000	1.4997180776663166	2.819223e-004	1.12e-02	0.0196

5. DISCUSSION OF RESULTS

We considered two real-life modeled first-order problems of the form (1) and a nonlinear problem. From the results obtained in the tables above, it is obvious that the quarter-step method derived is computationally reliable. The graphical results obtained also buttress the fact that the computed results converge toward the exact solution. We also discovered that the method developed in this paper performed better than that of the authors in [9]. It is also important to note that Laguerre polynomial was used as a basis function in the derivation of the Quarter-step method unlike the conventional power series usually used, see [11]. Thus, we may say that the higher the number of off-grid points, order of the method and the degree of the basis polynomial, the better the result.

6. CONCLUSION

Conclusively, a quarter-step method for the integration of modeled first-order problems of the form (1) using Laguerre polynomial of degree six as our basis function was developed. The method developed was found to be A-stable and that explained why it performed well on the class of problems it was applied on. The method was also found to be zero-stable, consistent, convergent and computationally reliable. We therefore recommend this method for the integration of first-order modeled problems of the form (1).

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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