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# A Study on Generalized Blaise Numbers

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***Author's contribution***

*The sole author designed, analyzed, interpreted and prepared the manuscript.*

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## ABSTRACT

In this paper, we introduce and investigate the generalized Blaise sequences and we deal with, in detail, two special cases, namely, Blaise and Blaise-Lucas sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Furthermore, we show that there are close relations between Blaise, Blaise-Lucas and Jacobsthal-Padovan, Jacobsthal-Perrin, adjusted Jacobsthal-Padovan, modified Jacobsthal-Padovan numbers. Moreover, we give some identities and matrices related with these sequences.

*Keywords: Blaise numbers; Blaise-Lucas numbers; Jacobsthal-Padovan numbers; Jacobsthal-Perrin numbers; adjusted Jacobsthal-Padovan numbers; modified Jacobsthal-Padovan numbers.*

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## 1 INTRODUCTION

Adjusted Jacobsthal-Padovan sequence $\{K_n\}_{n\geq 0}$ (OEIS: A159287, [1]), Jacobsthal-Perrin (Jacobsthal-Perrin-Lucas) sequence $\{L_n\}_{n\geq 0}$ (OEIS: A072328, [1]), Jacobsthal-Padovan sequence $\{Q_n\}_{n\geq 0}$ (OEIS: A159284, [1]), and modified Jacobsthal-Padovan sequence $\{M_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

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$$\begin{aligned}
K_{n+3} &= K_{n+1}+2K_n, \quad K_0=0, K_1=1, K_2=0, && (1.1)\\
L_{n+3} &= L_{n+1}+2L_n, \quad L_0=3, L_1=0, L_2=2, && (1.2)\\
Q_{n+3} &= Q_{n+1}+2Q_n, \quad Q_0=1, Q_1=1, Q_2=1, && (1.3)\\
M_{n+3} &= M_{n+1}+2M_n, \quad M_0=3, M_1=1, M_2=3. && (1.4)
\end{aligned}$$

The sequences $\{Q_n\}_{n\geq 0}$, $\{L_n\}_{n\geq 0}$, $\{K_n\}_{n\geq 0}$ and $\{M_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned}
K_{-n} &= -\frac{1}{2}K_{-(n-1)}+\frac{1}{2}K_{-(n-3)},\\
L_{-n} &= -\frac{1}{2}L_{-(n-1)}+\frac{1}{2}L_{-(n-3)},\\
Q_{-n} &= -\frac{1}{2}Q_{-(n-1)}+\frac{1}{2}Q_{-(n-3)},\\
M_{-n} &= -\frac{1}{2}M_{-(n-1)}+\frac{1}{2}M_{-(n-3)},
\end{aligned}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences (1.1)-(1.4) hold for all integer $n.$ For more information on Jacobsthal-Padovan sequence, see [2] and [3].

Now, we define two sequences related to Adjusted Jacobsthal-Padovan, Jacobsthal-Perrin (Jacobsthal-Perrin-Lucas), Jacobsthal-Padovan, and modified Jacobsthal-Padovan numbers. Blaise and Blaise-Lucas numbers are defined as

$$B_n = B_{n-2}+2B_{n-3}+1, \quad \text{with } B_0=0, B_1=1, B_2=1, \quad n\geq 3,$$

and

$$C_n = C_{n-2}+2C_{n-3}-2, \quad \text{with } C_0=4, C_1=1, C_2=3, \quad n\geq 3,$$

respectively. The first few values of Blaise and Blaise-Lucas numbers are

$$0,1,1,2,4,5,9,14,20,33,49,74,116,173,...$$

and

$$4,1,3,7,3,11,15,15,35,43,63,111,147,235,...$$

respectively. The sequences $\{B_n\}$ and $\{C_n\}$ satisfy the following fourth order linear recurrences:

$$\begin{aligned}
B_n &= B_{n-1}+B_{n-2}+B_{n-3}-2B_{n-4}, \quad B_0=0, B_1=1, B_2=1, B_3=2, \quad n\geq 4,\\
C_n &= C_{n-1}+C_{n-2}+C_{n-3}-2C_{n-4}, \quad C_0=4, C_1=1, C_2=3, C_3=7, \quad n\geq 4.
\end{aligned}$$

There are close relations between Blaise, Blaise-Lucas and Adjusted Jacobsthal-Padovan, Jacobsthal-Perrin (Jacobsthal-Perrin-Lucas), Jacobsthal-Padovan, and modified Jacobsthal-Padovan numbers. For example, they satisfy the following interrelations:

$$\begin{aligned}
2B_n &= K_{n+2}+K_{n+1}+2K_n-1,\\
2C_n &= -K_{n+2}+6K_{n+1}+K_n+2,\\
52B_n &= 2L_n+9L_{n+1}+10L_{n+2}-26,\\
C_n &= L_n+1,\\
2B_n &= Q_{n+2}-1,\\
2C_n &= -3Q_{n+2}+2Q_{n+1}+7Q_n+2,\\
46B_n &= 8M_{n+2}+5M_{n+1}-2M_n-23,\\
46C_n &= -5M_{n+2}-6M_{n+1}+53M_n+46,
\end{aligned}$$

and

$$\begin{aligned}
2K_n &= -B_{n+2}+3B_n+1, \\
52K_n &= 9C_{n+2}-C_{n+1}-6C_n-2, \\
4L_n &= -3B_{n+2}+14B_{n+1}-9B_n+1, \\
2L_n &= C_{n+3}-C_{n+1}, \\
2Q_n &= -B_{n+2}+2B_{n+1}+B_n+1, \\
26Q_n &= 4C_{n+2}+C_{n+1}+6C_n-11, \\
2M_n &= -B_{n+2}+6B_{n+1}-3B_n+1, \\
26M_n &= 3C_{n+2}+4C_{n+1}+24C_n-31.
\end{aligned}$$

The purpose of this article is to generalize and investigate these interesting sequence of numbers (i.e., Blaise, Blaise-Lucas numbers). First, we recall some properties of the generalized Tetranacci numbers.

The generalized $(r,s,t,u)$ sequence (or generalized Tetranacci sequence or generalized $4$-step Fibonacci sequence) $\{W_n(W_0,W_1,W_2,W_3;r,s,t,u)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1}+sW_{n-2}+tW_{n-3}+uW_{n-4}, \qquad W_0=c_0, W_1=c_1, W_2=c_2, W_3=c_3, \ \ n\geq 4 \tag{1.5}$$

where $W_0, W_1, W_2, W_3$ are arbitrary complex (or real) numbers and $r, s, t, u$ are real numbers.

This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [4,5,6,7,8,9,10,11,12]. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{t}{u}W_{-(n-1)} - \frac{s}{u}W_{-(n-2)} - \frac{r}{u}W_{-(n-3)} + \frac{1}{u}W_{-(n-4)}$$

for $n = 1, 2, 3, ...$ when $u \neq 0$. Therefore, recurrence (1.5) holds for all integers $n$.

As $\{W_n\}$ is a fourth-order recurrence sequence (difference equation), its characteristic equation is

$$z^4 - rz^3 - sz^2 - tz - u = 0 \tag{1.6}$$

whose roots are $\alpha, \beta, \gamma, \delta$. Note that we have the following identities

$$\begin{aligned}
\alpha+\beta+\gamma+\delta &= r, \\
\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta &= -s, \\
\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta &= t, \\
\alpha\beta\gamma\delta &= -u.
\end{aligned}$$

Using these roots and the recurrence relation, Binet's formula can be given as follows:

**Theorem 1.1.** *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta$) For all integers $n$, Binet's formula of generalized Tetranacci numbers is*

$$W_n = \frac{p_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{p_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{p_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{p_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \tag{1.7}$$

*where*

$$\begin{aligned}
p_1 &= W_3-(\beta+\gamma+\delta)W_2+(\beta\gamma+\beta\delta+\gamma\delta)W_1-\beta\gamma\delta W_0, \\
p_2 &= W_3-(\alpha+\gamma+\delta)W_2+(\alpha\gamma+\alpha\delta+\gamma\delta)W_1-\alpha\gamma\delta W_0, \\
p_3 &= W_3-(\alpha+\beta+\delta)W_2+(\alpha\beta+\alpha\delta+\beta\delta)W_1-\alpha\beta\delta W_0, \\
p_4 &= W_3-(\alpha+\beta+\gamma)W_2+(\alpha\beta+\alpha\gamma+\beta\gamma)W_1-\alpha\beta\gamma W_0.
\end{aligned}$$

Usually, it is customary to choose $\alpha, \beta, \gamma, \delta$ so that the Equ. (1.6) has at least one real (say $\alpha$) solutions. Note that the Binet form of a sequence satisfying (1.6) for non-negative integers is valid for all integers $n$ (see [13]).

Next, we consider two special cases of the generalized $(r,s,t,u)$ sequence $\{W_n\}$ which we call them $(r,s,t,u)$-Fibonacci and $(r,s,t,u)$-Lucas sequences. $(r,s,t,u)$-Fibonacci sequence $\{G_n\}_{n\geq 0}$ and $(r,s,t,u)$-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations

$$\begin{aligned} G_{n+4} &= rG_{n+3} + sG_{n+2} + tG_{n+1} + uG_n, \qquad (1.8)\\ G_0 &= 0, G_1 = 1, G_2 = r, G_3 = r^2 + s, \\ H_{n+4} &= rH_{n+3} + sH_{n+2} + tH_{n+1} + uH_n, \qquad (1.9)\\ H_0 &= 4, H_1 = r, H_2 = 2s + r^2, H_3 = r^3 + 3sr + 3t. \end{aligned}$$

The sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{t}{u}G_{-(n-1)} - \frac{s}{u}G_{-(n-2)} - \frac{r}{u}G_{-(n-3)} + \frac{1}{u}G_{-(n-4)}, \\ H_{-n} &= -\frac{t}{u}H_{-(n-1)} - \frac{s}{u}H_{-(n-2)} - \frac{r}{u}H_{-(n-3)} + \frac{1}{u}H_{-(n-4)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.8) and (1.9) hold for all integers $n$.

For all integers $n$, $(r,s,t,u)$-Fibonacci and $(r,s,t,u)$-Lucas numbers (using initial conditions in (1.8) or (1.9)) can be expressed using Binet's formulas as in the following corollary.

**Corollary 1.2.** *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta$) Binet's formula of $(r,s,t,u)$-Fibonacci and $(r,s,t,u)$-Lucas numbers are*

$$G_n = \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

*and*

$$H_n = \alpha^n + \beta^n + \gamma^n + \delta^n,$$

*respectively.*

Proof. Take $W_n = G_n$ and $W_n = H_n$ in Theorem 1.1, respectively. □

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} W_n z^n$ of the sequence $W_n$.

**Lemma 1.3.** *Suppose that $f_{W_n}(z) = \sum\limits_{n=0}^{\infty} W_n z^n$ is the ordinary generating function of the generalized $(r,s,t,u)$ sequence $\{W_n\}_{n\geq 0}$. Then, $\sum\limits_{n=0}^{\infty} W_n z^n$ is given by*

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - rW_0)z + (W_2 - rW_1 - sW_0)z^2 + (W_3 - rW_2 - sW_1 - tW_0)z^3}{1 - rz - sz^2 - tz^3 - uz^4}. \qquad (1.10)$$

Proof. For a proof, see Soykan [8, Lemma 1]. □

The following theorem presents Simson's formula of generalized $(r,s,t,u)$ sequence (generalized Tetranacci sequence) $\{W_n\}$.

**Theorem 1.4** (Simson's Formula of Generalized $(r,s,t,u)$ Numbers)**.** *For all integers $n$, we have*

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = (-1)^n u^n \begin{vmatrix} W_3 & W_2 & W_1 & W_0 \\ W_2 & W_1 & W_0 & W_{-1} \\ W_1 & W_0 & W_{-1} & W_{-2} \\ W_0 & W_{-1} & W_{-2} & W_{-3} \end{vmatrix}. \qquad (1.11)$$

*Proof.* (1.11) is given in Soykan [14]. □

The following theorem shows that the generalized Tetranacci sequence $W_n$ at negative indices can be expressed by the sequence itself at positive indices.

**Theorem 1.5.** *For $n \in \mathbb{Z}$, for the generalized Tetranacci sequence (or generalized $(r,s,t,u)$-sequence or 4-step Fibonacci sequence) we have the following:*

$$\begin{aligned} W_{-n} &= \frac{1}{6}(-u)^{-n}(-6W_{3n} + 6H_nW_{2n} - 3H_n^2W_n + 3H_{2n}W_n + W_0H_n^3 + 2W_0H_{3n} - 3W_0H_nH_{2n}) \\ &= (-1)^{-n-1}u^{-n}(W_{3n} - H_nW_{2n} + \frac{1}{2}(H_n^2 - H_{2n})W_n - \frac{1}{6}(H_n^3 + 2H_{3n} - 3H_{2n}H_n)W_0). \end{aligned}$$

Proof. For the proof, see Soykan [15, Theorem 1.]. □

Using Theorem 1.5, we have the following corollary, see Soykan [15, Corollary 4].

**Corollary 1.6.** *For $n \in \mathbb{Z}$, we have*

**(a)** $2\,(-u)^{n+4}\,G_{-n} = -(3ru^2+t^3-3stu)^2G_n^3 - (2su-t^2)^2G_{n+3}^2G_n - (-rt^2-tu+2rsu)^2G_{n+2}^2G_n - (-st^2+2s^2u+4u^2+rtu)^2G_{n+1}^2G_n + 2(3ru^2+t^3-3stu)((-2su+t^2)G_{n+3} + (-rt^2-tu+2rsu)G_{n+2} + (-st^2+2s^2u+4u^2+rtu)G_{n+1})G_n^2 + 2(2su-t^2)(-rt^2-tu+2rsu)G_{n+3}G_{n+2}G_n + 2(2su-t^2)(-st^2+2s^2u+4u^2+rtu)G_{n+3}G_{n+1}G_n - 2(-st^2+2s^2u+4u^2+rtu)(-rt^2-tu+2rsu)G_{n+2}G_{n+1}G_n - 2G_{3n}u^4 + u^2(-2su+t^2)G_{2n+3}G_n + u^2(-rt^2-tu+2rsu)G_{2n+2}G_n + u^2(-st^2+2s^2u+4u^2+rtu)G_{2n+1}G_n - 2u^2(2su-t^2)G_{2n}G_{n+3} + 2u^2(-rt^2-tu+2rsu)G_{2n}G_{n+2} + 2u^2(-st^2+2s^2u+4u^2+rtu)G_{2n}G_{n+1} - 3u^2(3ru^2+t^3-3stu)G_{2n}G_n.$

**(b)** $H_{-n} = \frac{1}{6}\,(-u)^{-n}\left(H_n^3 + 2H_{3n} - 3H_{2n}H_n\right).$

Note that $G_{-n}$ and $H_{-n}$ can be given as follows by using $G_0 = 0$ and $H_0 = 4$ in Theorem 1.5,

$$\begin{aligned} G_{-n} &= \frac{1}{6}(-u)^{-n}(-6G_{3n} + 6H_nG_{2n} - 3H_n^2G_n + 3H_{2n}G_n), && (1.12) \\ H_{-n} &= \frac{1}{6}\,(-u)^{-n}\left(H_n^3 + 2H_{3n} - 3H_{2n}H_n\right), && (1.13) \end{aligned}$$

respectively.

If we define the square matrix $A$ of order $4$ as

$$A = A_{rstu} = \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and also define

$$B_n = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} + uG_{n-2} & tG_n + uG_{n-1} & uG_n \\ G_n & sG_{n-1} + tG_{n-2} + uG_{n-3} & tG_{n-1} + uG_{n-2} & uG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} + uG_{n-4} & tG_{n-2} + uG_{n-3} & uG_{n-2} \\ G_{n-2} & sG_{n-3} + tG_{n-4} + uG_{n-5} & tG_{n-3} + uG_{n-4} & uG_{n-3} \end{pmatrix}$$

and

$$U_n = \begin{pmatrix} W_{n+1} & sW_n + tW_{n-1} + uW_{n-2} & tW_n + uW_{n-1} & uW_n \\ W_n & sW_{n-1} + tW_{n-2} + uW_{n-3} & tW_{n-1} + uW_{n-2} & uW_{n-1} \\ W_{n-1} & sW_{n-2} + tW_{n-3} + uW_{n-4} & tW_{n-2} + uW_{n-3} & uW_{n-2} \\ W_{n-2} & sW_{n-3} + tW_{n-4} + uW_{n-5} & tW_{n-3} + uW_{n-4} & uW_{n-3} \end{pmatrix}$$

then we get the following Theorem.

**Theorem 1.7.** *For all integers $m, n,$ we have*

**(a)** $B_n = A^n$, *i.e.,*

$$\begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & sG_n + tG_{n-1} + uG_{n-2} & tG_n + uG_{n-1} & uG_n \\ G_n & sG_{n-1} + tG_{n-2} + uG_{n-3} & tG_{n-1} + uG_{n-2} & uG_{n-1} \\ G_{n-1} & sG_{n-2} + tG_{n-3} + uG_{n-4} & tG_{n-2} + uG_{n-3} & uG_{n-2} \\ G_{n-2} & sG_{n-3} + tG_{n-4} + uG_{n-5} & tG_{n-3} + uG_{n-4} & uG_{n-3} \end{pmatrix}.$$

**(b)** $U_1 A^n = A^n U_1$.

**(c)** $U_{n+m} = U_n B_m = B_m U_n$.

**Proof.** For the proof, see Soykan [8, Theorem 19]. □

**Theorem 1.8.** *For all integers $m, n,$ we have*

$$W_{n+m} = W_n G_{m+1} + W_{n-1}(sG_m + tG_{m-1} + uG_{m-2}) + W_{n-2}(tG_m + uG_{m-1}) + uW_{n-3}G_m. \tag{1.14}$$

**Proof.** For the proof, see Soykan [8, Theorem 20]. □

In the next sections, we present new results.

# 2 GENERALIZED BLAISE SEQUENCE

In this paper, we consider the case $r = 1, s = 1, t = 1, u = -2$. A generalized Blaise sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relation

$$W_n = W_{n-1} + W_{n-2} + W_{n-3} - 2W_{n-4} \tag{2.1}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = \frac{1}{2}W_{-(n-1)} + \frac{1}{2}W_{-(n-2)} + \frac{1}{2}W_{-(n-3)} - \frac{1}{2}W_{-(n-4)}$$

for $n = 1, 2, 3, ....$ Therefore, recurrence (2.1) holds for all integers $n$.

Characteristic equation of $\{W_n\}$ is

$$z^4 - z^3 - z^2 - z + 2 = (z^3 - z - 2)(z - 1) = 0$$

whose roots are

$$\begin{aligned} \alpha &= \sqrt[3]{1 + \frac{\sqrt{78}}{9}} + \sqrt[3]{1 - \frac{\sqrt{78}}{9}} \simeq 1.521379706804568, \\ \beta &= \omega\sqrt[3]{1 + \frac{\sqrt{78}}{9}} + \omega^2\sqrt[3]{1 - \frac{\sqrt{78}}{9}}, \\ \gamma &= \omega^2\sqrt[3]{1 + \frac{\sqrt{78}}{9}} + \omega\sqrt[3]{1 - \frac{\sqrt{78}}{9}}, \\ \delta &= 1, \end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\begin{aligned} \alpha + \beta + \gamma + \delta &= 1, \\ \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= -1, \\ \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= 1, \\ \alpha\beta\gamma\delta &= 2. \end{aligned}$$

Note also that

$$\begin{aligned} \alpha + \beta + \gamma &= 0, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= -1, \\ \alpha\beta\gamma &= 2. \end{aligned}$$

The first few generalized Blaise numbers with positive subscript and negative subscript are given in the following Table 1.

**Table 1. A few generalized Blaise numbers**

| $n$ | $W_n$ | $W_{-n}$ |
|---|---|---|
| 0 | $W_0$ | $W_0$ |
| 1 | $W_1$ | $\frac{1}{2}(W_0 + W_1 + W_2 - W_3)$ |
| 2 | $W_2$ | $\frac{1}{4}(3W_0 + 3W_1 - W_2 - W_3)$ |
| 3 | $W_3$ | $\frac{1}{8}(9W_0 + W_1 + W_2 - 3W_3)$ |
| 4 | $W_1 - 2W_0 + W_2 + W_3$ | $\frac{1}{16}(11W_0 + 11W_1 + 3W_2 - 9W_3)$ |
| 5 | $2W_2 - W_1 - 2W_0 + 2W_3$ | $\frac{1}{32}(33W_0 + 17W_1 - 7W_2 - 11W_3)$ |
| 6 | $W_2 - 4W_0 + 4W_3$ | $\frac{1}{64}(67W_0 + 19W_1 + 11W_2 - 33W_3)$ |
| 7 | $4W_2 - 8W_0 + 5W_3$ | $\frac{1}{128}(105W_0 + 89W_1 + W_2 - 67W_3)$ |
| 8 | $5W_2 - 3W_1 - 10W_0 + 9W_3$ | $\frac{1}{256}(283W_0 + 107W_1 - 29W_2 - 105W_3)$ |
| 9 | $6W_2 - W_1 - 18W_0 + 14W_3$ | $\frac{1}{512}(497W_0 + 225W_1 + 73W_2 - 283W_3)$ |
| 10 | $13W_2 - 4W_1 - 28W_0 + 20W_3$ | $\frac{1}{1024}(947W_0 + 643W_1 - 69W_2 - 497W_3)$ |
| 11 | $16W_2 - 8W_1 - 40W_0 + 33W_3$ | $\frac{1}{2048}(2233W_0 + 809W_1 - 47W_2 - 947W_3)$ |
| 12 | $25W_2 - 7W_1 - 66W_0 + 49W_3$ | $\frac{1}{4096}(3851W_0 + 2139W_1 + 339W_2 - 2233W_3)$ |
| 13 | $42W_2 - 17W_1 - 98W_0 + 74W_3$ | $\frac{1}{8192}(8129W_0 + 4529W_1 - 615W_2 - 3851W_3)$ |

*Note that the sequences $\{B_n\}$ and $\{C_n\}$ which are defined in the section Introduction, are the special cases of the generalized Blaise sequence $\{W_n\}$. For convenience, we can give the definition of these two special cases of the sequence $\{W_n\}$ in this section as well. Blaise sequence $\{B_n\}_{n\geq 0}$ and Blaise-Lucas sequence $\{C_n\}_{n\geq 0}$ are defined, respectively, by the fourth-order recurrence relations*

$$\begin{aligned} B_n &= B_{n-1} + B_{n-2} + B_{n-3} - 2B_{n-4}, \quad B_0 = 0, B_1 = 1, B_2 = 1, B_3 = 2, \quad n \geq 4, \\ C_n &= C_{n-1} + C_{n-2} + C_{n-3} - 2C_{n-4}, \quad C_0 = 4, C_1 = 1, C_2 = 3, C_3 = 7, \quad n \geq 4. \end{aligned}$$

The sequences $\{B_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} B_{-n} &= \frac{1}{2}B_{-(n-1)} + \frac{1}{2}B_{-(n-2)} + \frac{1}{2}B_{-(n-3)} - \frac{1}{2}B_{-(n-4)} \\ C_{-n} &= \frac{1}{2}C_{-(n-1)} + \frac{1}{2}C_{-(n-2)} + \frac{1}{2}C_{-(n-3)} - \frac{1}{2}C_{-(n-4)} \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively.

Next, we present the first few values of the Blaise and Blaise-Lucas numbers with positive and negative subscripts:

**Table 2. The first few values of the special third-order numbers with positive and negative subscripts**

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $B_n$ | 0 | 1 | 1 | 2 | 4 | 5 | 9 | 14 | 20 | 33 | 49 | 74 | 116 | 173 |
| $B_{-n}$ | 0 | 0 | 0 | $-\frac{1}{2}$ | $-\frac{1}{4}$ | $-\frac{3}{8}$ | $-\frac{9}{16}$ | $-\frac{11}{32}$ | $-\frac{33}{64}$ | $-\frac{67}{128}$ | $-\frac{105}{256}$ | $-\frac{283}{512}$ | $-\frac{497}{1024}$ | $-\frac{947}{2048}$ |
| $C_n$ | 4 | 1 | 3 | 7 | 3 | 11 | 15 | 15 | 35 | 43 | 63 | 111 | 147 | 235 |
| $C_{-n}$ | 4 | $\frac{1}{2}$ | $\frac{5}{4}$ | $\frac{19}{8}$ | $\frac{1}{16}$ | $\frac{51}{32}$ | $\frac{89}{64}$ | $\frac{43}{128}$ | $\frac{417}{256}$ | $\frac{451}{512}$ | $\frac{745}{1024}$ | $\frac{2971}{2048}$ | $\frac{2929}{4096}$ | $\frac{8243}{8192}$ |

Theorem 1.1 can be used to obtain the Binet formula of generalized Blaise numbers. Using these (the above) roots and the recurrence relation, Binet's formula of generalized Blaise numbers can be given as follows:

**Theorem 2.1.** *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta = 1$) For all integers $n$, Binet's formula of generalized Blaise numbers is*

$$\begin{aligned} W_n &= \frac{(\alpha W_3 - \alpha(1-\alpha)W_2 + (-\alpha^2+2)W_1 - 2W_0)\alpha^n}{2\alpha^2+4\alpha-6} + \frac{(\beta W_3 - \beta(1-\beta)W_2 + (-\beta^2+2)W_1 - 2W_0)\beta^n}{2\beta^2+4\beta-6} \\ &\quad + \frac{(\gamma W_3 - \gamma(1-\gamma)W_2 + (-\gamma^2+2)W_1 - 2W_0)\gamma^n}{2\gamma^2+4\gamma-6} - \frac{W_3 - W_1 - 2W_0}{2} \end{aligned}$$

Blaise and Blaise-Lucas numbers can be expressed using Binet's formulas as follows:

**Corollary 2.2.** *(Four Distinct Roots Case: $\alpha \neq \beta \neq \gamma \neq \delta = 1$) For all integers $n$, Binet's formulas of Blaise and Blaise-Lucas numbers are*

$$B_n = \frac{\alpha^{n+3}}{2\alpha^2+4\alpha-6} + \frac{\beta^{n+3}}{2\beta^2+4\beta-6} + \frac{\gamma^{n+3}}{2\gamma^2+4\gamma-6} - \frac{1}{2}$$

*and*

$$C_n = \alpha^n + \beta^n + \gamma^n + 1$$

*respectively.*

Note that for all integers $n$, adjusted Jacobsthal-Padovan, Jacobsthal-Perrin (Jacobsthal-Perrin-Lucas), Jacobsthal-Padovan, and modified Jacobsthal-Padovan numbers can be expressed using Binet's formulas as

$$\begin{aligned} K_n &= \frac{1}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{n+1} + \frac{1}{(\beta-\alpha)(\beta-\gamma)}\beta^{n+1} + \frac{1}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{n+1}, \\ L_n &= \alpha^n + \beta^n + \gamma^n, \\ M_n &= \frac{(3\alpha+1)}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{n+1} + \frac{(3\beta+1)}{(\beta-\alpha)(\beta-\gamma)}\beta^{n+1} + \frac{(3\gamma+1)}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{n+1}, \\ Q_n &= \frac{(\alpha+1)}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{n+1} + \frac{(\beta+1)}{(\beta-\alpha)(\beta-\gamma)}\beta^{n+1} + \frac{(\gamma+1)}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{n+1}, \end{aligned}$$

respectively, see Soykan [2] for more details. So, by using Binet's formulas of Blaise, Blaise-Lucas and Adjusted Jacobsthal-Padovan, Jacobsthal-Perrin (Jacobsthal-Perrin-Lucas), Jacobsthal-Padovan, and modified Jacobsthal-Padovan numbers, (or by using mathematical induction), we get the following Lemma which contains many identities:

**Lemma 2.3.** *For all integers $n$, the following equalities (identities) are true:*

**(a)**

- $K_{n+1} = B_{n+1} - B_n.$

- $2K_n = B_{n+3} - B_{n+2} - B_{n+1} + B_n.$
- $2B_{n+4} = 5K_{n+2} + 9K_{n+1} + 6K_n - 1.$
- $2B_n = K_{n+2} + K_{n+1} + 2K_n - 1.$
- $2K_n = -B_{n+2} + 3B_n + 1.$

**(b)**

- $52K_{n+3} = 12C_{n+3} + 17C_{n+2} - 11C_{n+1} - 18C_n.$
- $52K_n = C_{n+3} - 2C_{n+1} + 9C_{n+2} - 8C_n.$
- $C_{n+4} = 6K_{n+2} + 2K_{n+1} + 4K_n + 1.$
- $2C_n = -K_{n+2} + 6K_{n+1} + K_n + 2.$
- $52K_n = 9C_{n+2} - C_{n+1} - 6C_n - 2.$
- $C_{n+1} + 6C_n = 18K_{n+1} + 2K_n + 7.$

**(c)**

- $L_{n+3} = 2B_{n+2} + 4B_{n+1} - 6B_n.$
- $4L_n = B_{n+3} - 3B_{n+2} + 13B_{n+1} - 11B_n.$
- $52B_{n+4} = 30L_{n+2} + 53L_{n+1} + 58L_n - 26.$
- $52B_n = 2L_n + 9L_{n+1} + 10L_{n+2} - 26.$
- $4L_n = -3B_{n+2} + 14B_{n+1} - 9B_n + 1.$
- $4(10B_{n+1} - 9B_n) = 14L_n + 3L_{n+1} - 2.$

**(d)**

- $2L_{n+3} = 3C_{n+3} - C_{n+1} - 2C_n.$
- $2L_n = C_{n+3} - C_{n+1}.$
- $C_{n+4} = L_{n+2} + 2L_{n+1} + 1.$
- $C_n = L_n + 1.$
- $L_n = C_n - 1.$

**(e)**

- $Q_{n+3} = B_{n+3} + B_{n+1} - 2B_n.$
- $2Q_n = B_{n+3} - B_{n+2} + B_{n+1} - B_n.$
- $2B_{n+4} = Q_{n+2} + 4Q_{n+1} + 4Q_n - 1.$
- $2B_n = Q_{n+2} - 1.$
- $2Q_n = -B_{n+2} + 2B_{n+1} + B_n + 1.$
- $2B_{n+1} = Q_{n+1} + 2Q_n - 1.$

**(f)**

- $52Q_{n+3} = 41C_{n+3} + 18C_{n+2} - 17C_{n+1} - 42C_n.$
- $52Q_n = 11C_{n+3} + 8C_{n+2} - 9C_{n+1} - 10C_n.$
- $C_{n+4} = 4Q_{n+2} + 2Q_{n+1} - 4Q_n + 1.$
- $2C_n = -3Q_{n+2} + 2Q_{n+1} + 7Q_n + 2.$
- $26Q_n = 4C_{n+2} + C_{n+1} + 6C_n - 11.$
- $3C_{n+1} + 2C_n = 8Q_{n+1} - 2Q_n + 5.$

**(g)**

- $M_{n+3} = B_{n+3} + 2B_{n+2} + 3B_{n+1} - 6B_n$.
- $2M_n = B_{n+3} - B_{n+2} + 5B_{n+1} - 5B_n$.
- $46B_{n+4} = 16M_{n+2} + 33M_{n+1} + 42M_n - 23$.
- $46B_n = 8M_{n+2} + 5M_{n+1} - 2M_n - 23$.
- $2M_n = -B_{n+2} + 6B_{n+1} - 3B_n + 1$.
- $2(8B_{n+1} - 5B_n) = M_{n+1} + 6M_n - 3$.

**(h)**

- $52M_{n+3} = 99C_{n+3} + 20C_{n+2} - 29C_{n+1} - 90C_n$.
- $52M_n = 31C_{n+3} + 6C_{n+2} - 23C_{n+1} - 14C_n$.
- $23C_{n+4} = 18M_{n+2} + 40M_{n+1} - 16M_n + 23$.
- $46C_n = -5M_{n+2} - 6M_{n+1} + 53M_n + 46$.
- $26M_n = 3C_{n+2} + 4C_{n+1} + 24C_n - 31$.
- $6C_n - 5C_{n+1} = 8M_n - 6M_{n+1} + 1$.

Proof. We only prove $K_{n+1} = B_{n+1} - B_n$ by using Binet's formulas of $K_n$ and $B_n$ as the others can be proved similarly. By using Binet's formulas, we get

$$\begin{aligned}
B_{n+1} - B_n &= \frac{\alpha^{n+4}}{2\alpha^2+4\alpha-6} - \frac{\alpha^{n+3}}{2\alpha^2+4\alpha-6} + \frac{\beta^{n+4}}{2\beta^2+4\beta-6} - \frac{\beta^{n+3}}{2\beta^2+4\beta-6} + \frac{\gamma^{n+4}}{2\gamma^2+4\gamma-6} - \frac{\gamma^{n+3}}{2\gamma^2+4\gamma-6} \\
&= \frac{1}{2}\frac{\alpha^{n+3}}{\alpha+3} + \frac{1}{2}\frac{\beta^{n+3}}{\beta+3} + \frac{1}{2}\frac{\gamma^{n+3}}{\gamma+3} \\
&= \frac{1}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{n+2} + \frac{1}{(\beta-\alpha)(\beta-\gamma)}\beta^{n+2} + \frac{1}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{n+2} \\
&= K_{n+1}
\end{aligned}$$

where

$$\begin{aligned}
\frac{1}{2}\frac{\alpha}{\alpha+3} &= \frac{1}{(\alpha-\beta)(\alpha-\gamma)}, \\
\frac{1}{2}\frac{\beta}{\beta+3} &= \frac{1}{(\beta-\alpha)(\beta-\gamma)}, \\
\frac{1}{2}\frac{\gamma}{\gamma+3} &= \frac{1}{(\gamma-\alpha)(\gamma-\beta)}. \quad \square
\end{aligned}$$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n z^n$ of the sequence $W_n$.

**Lemma 2.4.** *Suppose that $f_{W_n}(z) = \sum_{n=0}^{\infty} W_n z^n$ is the ordinary generating function of the generalized Blaise sequence $\{W_n\}$. Then, $\sum_{n=0}^{\infty} W_n z^n$ is given by*

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - W_0)z + (W_2 - W_1 - W_0)z^2 + (W_3 - W_2 - W_1 - W_0)z^3}{1 - z - z^2 - z^3 + 2z^4}.$$

Proof. Take $r=1, s=1, t=1, u=-2$ in Lemma 1.3.

The previous lemma gives the following results as particular examples.

**Corollary 2.5.** *Generating functions of Blaise and Blaise-Lucas numbers are*

$$\begin{aligned}\sum_{n=0}^{\infty} B_n z^n &= \frac{z}{1-z-z^2-z^3+2z^4},\\ \sum_{n=0}^{\infty} C_n z^n &= \frac{4-3z-2z^2-z^3}{1-z-z^2-z^3+2z^4},\end{aligned}$$

*respectively.*

Proof. Set $W_n = B_n,\ B_0 = 0, B_1 = 1, B_2 = 1, B_3 = 2$ and $W_n = C_n$, $C_0 = 4, C_1 = 1, C_2 = 3, C_3 = 7$ in the previous Lemma, respectively. □

# 3 SIMSON FORMULAS

Now, we present Simson's formula of generalized Blaise numbers.

**Theorem 3.1** (Simson's Formula of Generalized Blaise Numbers). *For all integers $n$, we have*

$$\begin{vmatrix} W_{n+3} & W_{n+2} & W_{n+1} & W_n \\ W_{n+2} & W_{n+1} & W_n & W_{n-1} \\ W_{n+1} & W_n & W_{n-1} & W_{n-2} \\ W_n & W_{n-1} & W_{n-2} & W_{n-3} \end{vmatrix} = 2^{n-3}(W_3 - W_1 - 2W_0)(W_3^3 + 2W_2^3 - W_1^3 - 4W_0^3 + (-3W_2 - W_1 + W_0)W_3^2 + 2(W_3 - W_1 - 3W_0)W_2^2 + (5W_3 + 3W_2 - 5W_0)W_1^2 + 4(W_2 + 2W_1)W_0^2 - 2W_1W_2W_3 + 4W_0W_2W_3 - 6W_0 W_1W_3).$$

Proof. Take $r=1, s=1, t=1, u=-2$ in Theorem 1.4. □
The previous theorem gives the following results as particular examples.

**Corollary 3.2.** *For all integers $n$, the Simson's formulas of Blaise and Blaise-Lucas numbers are given as*

$$\begin{aligned}\begin{vmatrix} B_{n+3} & B_{n+2} & B_{n+1} & B_n \\ B_{n+2} & B_{n+1} & B_n & B_{n-1} \\ B_{n+1} & B_n & B_{n-1} & B_{n-2} \\ B_n & B_{n-1} & B_{n-2} & B_{n-3} \end{vmatrix} &= 2^{n-1},\\ \begin{vmatrix} C_{n+3} & C_{n+2} & C_{n+1} & C_n \\ C_{n+2} & C_{n+1} & C_n & C_{n-1} \\ C_{n+1} & C_n & C_{n-1} & C_{n-2} \\ C_n & C_{n-1} & C_{n-2} & C_{n-3} \end{vmatrix} &= -13 \times 2^{n+2},\end{aligned}$$

*respectively.*

Proof. Set $W_n = B_n,\ B_0 = 0, B_1 = 1, B_2 = 1, B_3 = 2$ and $W_n = C_n$, $C_0 = 4, C_1 = 1, C_2 = 3, C_3 = 7$ in the previous Theorem, respectively. □

# 4 SOME IDENTITIES

In this section, we obtain some identities of Blaise and Blaise-Lucas numbers. First, we can give a few basic relations between $\{W_n\}$ and $\{B_n\}$.

**Lemma 4.1.** *The following equalities are true:*

**(a)** $16W_n = (11W_0 + 11W_1 + 3W_2 - 9W_3)B_{n+5} + (7W_0 - 9W_1 - W_2 + 3W_3)B_{n+4} - (17W_0 + W_1 + 9W_2 - 11W_3)B_{n+3} - (33W_0 + 17W_1 - 7W_2 - 11W_3)B_{n+2}.$

**(b)** $8W_n = (9W_0 + W_1 + W_2 - 3W_3)B_{n+4} - (3W_0 - 5W_1 + 3W_2 - W_3)B_{n+3} - (11W_0 + 3W_1 - 5W_2 - W_3)B_{n+2} - (11W_0 + 11W_1 + 3W_2 - 9W_3)B_{n+1}.$

**(c)** $4W_n = (3W_0 + 3W_1 - W_2 - W_3)B_{n+3} - (W_0 + W_1 - 3W_2 + W_3)B_{n+2} - (W_0 + 5W_1 + W_2 - 3W_3)B_{n+1} - (9W_0 + W_1 + W_2 - 3W_3)B_n.$

**(d)** $2W_n = (W_0 + W_1 + W_2 - W_3)B_{n+2} + (W_0 - W_1 - W_2 + W_3)B_{n+1} - (3W_0 - W_1 + W_2 - W_3)B_n - (3W_0 + 3W_1 - W_2 - W_3)B_{n-1}.$

**(e)** $W_n = W_0B_{n+1} + (W_1 - W_0)B_n + (W_2 - W_1 - W_0)B_{n-1} + (W_3 - W_1 - W_2 - W_0)B_{n-2}.$

Proof. Note that all the identities hold for all integers $n$. We prove (a). To show (a), writing

$$W_n = a \times B_{n+5} + b \times B_{n+4} + c \times B_{n+3} + d \times B_{n+2}$$

and solving the system of equations

$$\begin{array}{rcl} W_0 & = & a \times B_5 + b \times B_4 + c \times B_3 + d \times B_2 \\ W_1 & = & a \times B_6 + b \times B_5 + c \times B_4 + d \times B_3 \\ W_2 & = & a \times B_7 + b \times B_6 + c \times B_5 + d \times B_4 \\ W_3 & = & a \times B_8 + b \times B_7 + c \times B_6 + d \times B_5 \end{array}$$

we find that $16a = 11W_0+11W_1+3W_2-9W_3, 16b = 7W_0-9W_1-W_2+3W_3, 16c = 11W_3-W_1-9W_2-17W_0, 16d = 7W_2 - 17W_1 - 33W_0 + 11W_3$. The other equalities can be proved similarly. □

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{W_n\}$ and $\{C_n\}$.

**Lemma 4.2.** *The following equalities are true:*

**(a)** $104W_n = -(56W_0 + 27W_1 + 2W_2 - 33W_3)C_{n+5} - 2(5W_0 - 2W_1 - 4W_2 + W_3)C_{n+4} + (60W_0 + 15W_1 + 4W_2 - 27W_3)C_{n+3} + 2(55W_0 + 30W_1 - 5W_2 - 28W_3)C_{n+2}.$

**(b)** $104W_n = -(66W_0 + 23W_1 - 6W_2 - 31W_3)C_{n+4} + 2(2W_0 - 6W_1 + W_2 + 3W_3)C_{n+3} + (54W_0 + 33W_1 - 12W_2 - 23W_3)C_{n+2} + 2(56W_0 + 27W_1 + 2W_2 - 33W_3)C_{n+1}.$

**(c)** $104W_n = -(62W_0 + 35W_1 - 8W_2 - 37W_3)C_{n+3} - 2(6W_0 - 5W_1 + 3W_2 - 4W_3)C_{n+2} + (46W_0 + 31W_1 + 10W_2 - 35W_3)C_{n+1} + 2(66W_0 + 23W_1 - 6W_2 - 31W_3)C_n.$

**(d)** $104W_n = -(74W_0 + 25W_1 - 2W_2 - 45W_3)C_{n+2} - 2(8W_0 + 2W_1 - 9W_2 - W_3)C_{n+1} + (70W_0 + 11W_1 - 4W_2 - 25W_3)C_n + 2(62W_0 + 35W_1 - 8W_2 - 37W_3)C_{n-1}.$

**(e)** $104W_n = -(90W_0 + 29W_1 - 20W_2 - 47W_3)C_{n+1} - 2(2W_0 + 7W_1 + W_2 - 10W_3)C_n + (50W_0 + 45W_1 - 14W_2 - 29W_3)C_{n-1} + 2(74W_0 + 25W_1 - 2W_2 - 45W_3)C_{n-2}.$

Now, we give a few basic relations between $\{B_n\}$ and $\{C_n\}$.

**Lemma 4.3.** *The following equalities are true:*

$$\begin{aligned}
104B_n &= 37C_{n+5}+8C_{n+4}-35C_{n+3}-62C_{n+2},\\
104B_n &= 45C_{n+4}+2C_{n+3}-25C_{n+2}-74C_{n+1},\\
104B_n &= 47C_{n+3}+20C_{n+2}-29C_{n+1}-90C_n,\\
104B_n &= 67C_{n+2}+18C_{n+1}-43C_n-94C_{n-1},\\
104B_n &= 85C_{n+1}+24C_n-27C_{n-1}-134C_{n-2},
\end{aligned}$$

*and*

$$\begin{aligned}
16C_n &= B_{n+5}+37B_{n+4}-19B_{n+3}-51B_{n+2},\\
8C_n &= 19B_{n+4}-9B_{n+3}-25B_{n+2}-B_{n+1},\\
4C_n &= 5B_{n+3}-3B_{n+2}+9B_{n+1}-19B_n,\\
2C_n &= B_{n+2}+7B_{n+1}-7B_n-5B_{n-1},\\
C_n &= 4B_{n+1}-3B_n-2B_{n-1}-B_{n-2}.
\end{aligned}$$

# 5 RELATIONS BETWEEN SPECIAL NUMBERS

In this section, we present identities on Blaise, Blaise-Lucas numbers and adjusted Jacobsthal-Padovan, Jacobsthal-Perrin (Jacobsthal-Perrin-Lucas), Jacobsthal-Padovan, and modified Jacobsthal-Padovan numbers. We know from Lemma 2.3 that

$$\begin{aligned}
52B_n &= 2L_n+9L_{n+1}+10L_{n+2}-26,\\
2C_n &= -3Q_{n+2}+2Q_{n+1}+7Q_n+2.
\end{aligned}$$

Note also that from Lemma 4.1 and Lemma 4.2, we have the formulas of $W_n$ as

$$\begin{aligned}
4W_n &= (3W_0+3W_1-W_2-W_3)B_{n+3}-(W_0+W_1-3W_2+W_3)B_{n+2}\\
&\quad -(W_0+5W_1+W_2-3W_3)B_{n+1}-(9W_0+W_1+W_2-3W_3)B_n,\\
104W_n &= -(62W_0+35W_1-8W_2-37W_3)C_{n+3}-2(6W_0-5W_1+3W_2-4W_3)C_{n+2}\\
&\quad +(46W_0+31W_1+10W_2-35W_3)C_{n+1}+2(66W_0+23W_1-6W_2-31W_3)C_n.
\end{aligned}$$

Using the above identities, we obtain relation of generalized Blaise numbers in the following forms (in terms of Jacobsthal-Perrin (Jacobsthal-Perrin-Lucas) and Jacobsthal-Padovan numbers):

**Lemma 5.1.** *For all integers $n$, we have the following identities:*

**(a)** $52W_n=(4W_3-3W_2+5W_1-6W_0)L_{n+2}+(W_3+9W_2-2W_1-8W_0)L_{n+1}+2(3W_3+W_2-6W_1+2W_0)L_n-26W_3+26W_1+52W_0.$

**(b)** $2W_n=(W_1-W_0)Q_{n+2}+(W_2-W_1)Q_{n+1}+(W_3-W_2-W_1+W_0)Q_n-W_3+W_1+2W_0.$

# 6 ON THE RECURRENCE PROPERTIES OF GENERALIZED BLAISE SEQUENCE

Taking $r=1,s=1,t=1,u=-2$ in Theorem 1.5, we obtain the following Proposition.

**Proposition 6.1.** *For $n\in\mathbb{Z}$, generalized Blaise numbers (the case $r=1,s=1,t=1,u=-2$) have the following identity:*

$$W_{-n}=\frac{2^{-n-1}}{3}(-6W_{3n}+6C_nW_{2n}-3C_n^2W_n+3C_{2n}W_n+W_0C_n^3+2W_0C_{3n}-3W_0C_nC_{2n}).$$

From the above Proposition 6.1 (or by taking $G_n = B_n$ and $H_n = C_n$ in (1.12) and (1.13) respectively), we have the following corollary which gives the connection between the special cases of generalized Blaise sequence at the positive index and the negative index: for Blaise and Blaise-Lucas numbers: take $W_n = B_n$ with $B_0 = 0, B_1 = 1, B_2 = 1, B_3 = 2$ and take $W_n = C_n$ with $C_0 = 4, C_1 = 1, C_2 = 3, C_3 = 7$, respectively. Note that in this case $H_n = C_n$.

**Corollary 6.2.** *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

**(a)** *Blaise sequence:*

$$B_{-n} = \frac{2^{-n-1}}{3}(-6B_{3n} + 6C_nB_{2n} - 3C_n^2B_n + 3C_{2n}B_n).$$

**(b)** *Blaise-Lucas sequence:*

$$C_{-n} = \frac{2^{-n-1}}{3}(C_n^3 + 2C_{3n} - 3C_{2n}C_n).$$

We can also present the formulas of $B_{-n}$ and $C_{-n}$ in the following forms.

**Corollary 6.3.** *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

**(a)** $B_{-n} = \frac{2^{-n-5}}{3}(-96B_{3n} + 24(5B_{n+3} - 3B_{n+2} + 9B_{n+1} - 19B_n)B_{2n} - 3(5B_{n+3} - 3B_{n+2} + 9B_{n+1} - 19B_n)^2B_n + 12(5B_{2n+3} - 3B_{2n+2} + 9B_{2n+1} - 19B_{2n})B_n)$.

**(b)**

- **(i)** $2^{n+1}B_{-n} = -K_n^2 - K_{n-1}^2 - 2K_{n-2}^2 + (K_{n+2} - 6K_{n+1} - 6K_{n-1} + 2K_{n-2})K_n + (K_{n+1} - 12K_{n-2})K_{n-1} + 2K_{2n} + 2K_{2n-2} + 4K_{2n-4} - 2^n$.
- **(ii)** $2^{n+2}C_{-n} = -K_n^2 - 12K_{n-1}^2 + 4K_{n-2}^2 + (K_{n+2} - 6K_{n+1} - 72K_{n-1} - 4K_{n-2})K_n + 12(K_{n+1} + 2K_{n-2})K_{n-1} + 2K_{2n} + 24K_{2n-2} - 8K_{2n-4} + 2^{n+2}$.

**(c)**

- **(i)** $13B_{-n} = \frac{1}{2^{n+2}}(L_n^2 + 9L_{n-1}^2 + 20L_{n-2}^2 - L_{2n} - 9L_{2n-2} - 20L_{2n-4} - 13 \times 2^{n+1})$.
- **(ii)** $2^{n+1}C_{-n} = L_n^2 - L_{2n} + 2^{n+1}$.

**(d)**

- **(i)** $2^{n+2}B_{-n} = 9Q_n^2 + 4Q_{n-1}^2 + 21Q_{n-2}^2 + 6Q_{2n-2} - 4Q_{2n-3} - 6Q_{2n-4} - 12Q_nQ_{n-1} - 30Q_nQ_{n-2} + 20Q_{n-1}Q_{n-2} - 2^{n+1}$.
- **(ii)** $2^{n+4}C_{-n} = 63Q_{n+2}^2 + 64Q_{n+1}^2 + 55Q_n^2 + 36Q_{n-1}^2 - 252Q_{n-2}^2 + 2(-105Q_{n+2} + 46Q_{n+1} + 112Q_{n-1} + 180Q_{n-2})Q_n - 120(Q_{n+1} + 2Q_{n-2})Q_{n-1} - 84Q_{n+1}Q_{n+2} + 42Q_{2n+2} - 28Q_{2n+1} - 18Q_{2n} - 16Q_{2n-1} - 96Q_{2n-2} + 48Q_{2n-3} + 72Q_{2n-4} + 2^{n+4}$.

**(e)**

- **(i)** $12167B_{-n} = \frac{1}{2^{n+3}}(-75M_{n+2}^2 + 267M_{n+1}^2 - 1811M_n^2 + 19\,483M_{n-1}^2 + 56\,816M_{n-2}^2 + 2(+565M_{n+2} + 1128M_{n+1} - 1950M_{n-1} - 9040M_{n-2})M_n - 226(25M_{n+1} + 96M_{n-2})M_{n-1} - 180M_{n+1}M_{n+2} + 6532M_{2n} + 4140M_{2n-1} - 828M_{2n+1} - 4370M_{2n-2} - 690M_{2n+2} + 13\,248M_{2n-3} - 49\,312M_{2n-4} - 12167 \times 2^{n+2})$.
- **(ii)** $12167C_{-n} = \frac{1}{2^{n+4}}(3975M_{n+2}^2 + 4824M_{n+1}^2 + 185407M_n^2 - 44772M_{n-1}^2 - 71020M_{n-2}^2 + 2(-29945M_{n+2} - 37014M_{n+1} + 6336M_{n-1} + 11\,300M_{n-2})M_n + 13560(M_{n+1} + 2M_{n-2})M_{n-1} + 9540M_{n+1}M_{n+2} + 36570M_{2n+2} + 43884M_{2n+1} - 171626M_{2n} - 9936M_{2n-1} + 23184M_{2n-2} - 16560M_{2n-3} + 61640M_{2n-4} + 12167 \times 2^{n+4})$.

Proof. We use the identities, see Soykan [16],

$$\begin{aligned} K_{-n} &= \frac{1}{2^{n+1}}(-K_n^2+2K_{2n}+K_{n+2}K_n-6K_{n+1}K_n), \\ L_{-n} &= \frac{1}{2^{n+1}}(L_n^2-L_{2n}), \end{aligned}$$

and

$Q_{-n}=\frac{1}{2^{n+3}}(9Q_{n+2}^2+4Q_{n+1}^2+21Q_n^2+6Q_{2n+2}-4Q_{2n+1}-6Q_{2n}-12Q_{n+2}Q_{n+1}-30Q_{n+2}Q_n+20Q_{n+1}Q_n)$, $M_{-n}=\frac{1}{529\times 2^{n+3}}(108M_{n+1}^2+75M_{n+2}^2+3551M_n^2+690M_{2n+2}+828M_{2n+1}-3082M_{2n}+180M_{n+2}M_{n+1}-1130M_{n+2}M_n-1356M_{n+1}M_n)$.

We also use the identities

$$\begin{aligned} 2B_n &= K_{n+2}+K_{n+1}+2K_n-1, \\ 2C_n &= -K_{n+2}+6K_{n+1}+K_n+2, \\ 52B_n &= 2L_n+9L_{n+1}+10L_{n+2}-26, \\ C_n &= L_n+1, \\ 2B_n &= Q_{n+2}-1, \\ 2C_n &= -3Q_{n+2}+2Q_{n+1}+7Q_n+2, \\ 46B_n &= 8M_{n+2}+5M_{n+1}-2M_n-23, \\ 46C_n &= -5M_{n+2}-6M_{n+1}+53M_n+46. \end{aligned}$$

**(a)** By using the identity $4C_n=5B_{n+3}-3B_{n+2}+9B_{n+1}-19B_n$ and Corollary 4.3, (or by using Corollary 1.6 (a)), we obtain (a).

**(b)** Since $2B_n=K_{n+2}+K_{n+1}+2K_n-1$, $2C_n=-K_{n+2}+6K_{n+1}+K_n+2$ and $K_{-n}=\frac{1}{2^{n+1}}(-K_n^2+2K_{2n}+K_{n+2}K_n-6K_{n+1}K_n)$, we get (b)

**(c)** Since $52B_n=2L_n+9L_{n+1}+10L_{n+2}-26$, $C_n=L_n+1$ and $L_{-n}=\frac{1}{2^{n+1}}(L_n^2-L_{2n})$, we obtain (c).

**(d)** Since $2B_n=Q_{n+2}-1$, $2C_n=-3Q_{n+2}+2Q_{n+1}+7Q_n+2$ and $Q_{-n}=\frac{1}{2^{n+3}}(9Q_{n+2}^2+4Q_{n+1}^2+21Q_n^2+6Q_{2n+2}-4Q_{2n+1}-6Q_{2n}-12Q_{n+2}Q_{n+1}-30Q_{n+2}Q_n+20Q_{n+1}Q_n)$, we get (d).

**(e)** Since $46B_n=8M_{n+2}+5M_{n+1}-2M_n-23$, $46C_n=-5M_{n+2}-6M_{n+1}+53M_n+46$ and $M_{-n}=\frac{1}{529\times 2^{n+3}}(108M_{n+1}^2+75M_{n+2}^2+3551M_n^2+690M_{2n+2}+828M_{2n+1}-3082M_{2n}+180M_{n+2}M_{n+1}-1130M_{n+2}M_n-1356M_{n+1}M_n)$, we obtain (e). □

# 7 SUM FORMULAS

The following Corollary gives sum formulas of Jacobsthal-Padovan numbers.

**Corollary 7.1.** *For* $n\geq 0$ *we have the following formulas:*

**(a)** $\sum_{k=0}^{n}Q_k=\frac{1}{2}\left(Q_{n+3}+Q_{n+2}-2\right)$.

**(b)** $\sum_{k=0}^{n}Q_{2k}=\frac{1}{2}\left(Q_{2n+1}+2Q_{2n}-1\right)$.

**(c)** $\sum_{k=0}^{n}Q_{2k+1}=\frac{1}{2}\left(Q_{2n+2}+2Q_{2n+1}-1\right)$.

Proof. It is given in Soykan [2]. □

The following Corollary presents sum formulas of Blaise and Blaise-Lucas numbers.

**Corollary 7.2.** *For $n \geq 0,$ Blaise and Blaise-Lucas numbers have the following properties (in terms of Jacobsthal-Padovan numbers):*

**(a)**

- **(i)** $\sum_{k=0}^{n} B_k = \frac{1}{4}(3Q_{n+2} + 3Q_{n+1} + 2Q_n - 2(n+4)).$
- **(ii)** $\sum_{k=0}^{n} B_{2k} = \frac{1}{4}(2Q_{2n+2} + Q_{2n+1} + 2Q_{2n} - 2n - 5).$
- **(iii)** $\sum_{k=0}^{n} B_{2k+1} = \frac{1}{4}(Q_{2n+2} + 4Q_{2n+1} + 4Q_{2n} - 2n - 5).$

**(b)**

- **(i)** $\sum_{k=0}^{n} C_k = Q_{n+1} + 3Q_n + n.$
- **(ii)** $\sum_{k=0}^{n} C_{2k} = -Q_{2n+2} + 2Q_{2n+1} + 2Q_{2n} + n + 1.$
- **(iii)** $\sum_{k=0}^{n} C_{2k+1} = 2Q_{2n+2} + Q_{2n+1} - 2Q_{2n} + n.$

Proof. The proof follows from Corollary 7.1 and the identities

$$\begin{aligned} 2B_n &= Q_{n+2} - 1, \\ 2C_n &= -3Q_{n+2} + 2Q_{n+1} + 7Q_n + 2. \quad \square \end{aligned}$$

# 8 MATRICES AND IDENTITIES RELATED WITH GENERALIZED BLAISE NUMBERS

If we define the square matrix $A$ of order $4$ as

$$A = \begin{pmatrix} 1 & 1 & 1 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and also define

$$B_n = \begin{pmatrix} B_{n+1} & B_n + B_{n-1} - 2B_{n-2} & B_n - 2B_{n-1} & -2B_n \\ B_n & B_{n-1} + B_{n-2} - 2B_{n-3} & B_{n-1} - 2B_{n-2} & -2B_{n-1} \\ B_{n-1} & B_{n-2} + B_{n-3} - 2B_{n-4} & B_{n-2} - 2B_{n-3} & -2B_{n-2} \\ B_{n-2} & B_{n-3} + B_{n-4} - 2B_{n-5} & B_{n-3} - 2B_{n-4} & -2B_{n-3} \end{pmatrix}$$

and

$$U_n = \begin{pmatrix} W_{n+1} & W_n + W_{n-1} - 2W_{n-2} & W_n - 2W_{n-1} & -2W_n \\ W_n & W_{n-1} + W_{n-2} - 2W_{n-3} & W_{n-1} - 2W_{n-2} & -2W_{n-1} \\ W_{n-1} & W_{n-2} + W_{n-3} - 2W_{n-4} & W_{n-2} - 2W_{n-3} & -2W_{n-2} \\ W_{n-2} & W_{n-3} + W_{n-4} - 2W_{n-5} & W_{n-3} - 2W_{n-4} & -2W_{n-3} \end{pmatrix}.$$

then we get the following Theorem.

**Theorem 8.1.** *For all integers $m, n,$ we have*

**(a)** $B_n = A^n$, *i.e.,*

$$\begin{pmatrix} 1 & 1 & 1 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} B_{n+1} & B_n + B_{n-1} - 2B_{n-2} & B_n - 2B_{n-1} & -2B_n \\ B_n & B_{n-1} + B_{n-2} - 2B_{n-3} & B_{n-1} - 2B_{n-2} & -2B_{n-1} \\ B_{n-1} & B_{n-2} + B_{n-3} - 2B_{n-4} & B_{n-2} - 2B_{n-3} & -2B_{n-2} \\ B_{n-2} & B_{n-3} + B_{n-4} - 2B_{n-5} & B_{n-3} - 2B_{n-4} & -2B_{n-3} \end{pmatrix}.$$

**(b)** $U_1 A^n = A^n U_1$.

**(c)** $U_{n+m} = U_n B_m = B_m U_n$.

**Proof.** Take $r = 1, s = 1, t = 1, u = -2$ in Theorem 1.7. □

Using the above last Theorem and the identity

$$2B_n = Q_{n+2} - 1,$$

we obtain the following identity for Jacobsthal-Padovan numbers.

**Corollary 8.2.** *For all integers $n$, we have the following formula for Jacobsthal-Padovan numbers:*

$$A^n = \frac{1}{2}\begin{pmatrix} Q_{n+3}-1 & Q_{n+4}-Q_{n+3} & -Q_{n+4}+2Q_{n+2}+1 & -2Q_{n+2}+2 \\ Q_{n+2}-1 & Q_{n+3}-Q_{n+2} & -Q_{n+3}+2Q_{n+1}+1 & -2Q_{n+1}+2 \\ Q_{n+1}-1 & Q_{n+2}-Q_{n+1} & -Q_{n+2}+2Q_n+1 & -2Q_n+2 \\ Q_n-1 & Q_{n+1}-Q_n & -Q_{n+1}+2Q_{n-1}+1 & -2Q_{n-1}+2 \end{pmatrix}.$$

Next, we present an identity for $W_{n+m}$.

**Theorem 8.3.** *For all integers $m, n$, we have*

$$W_{n+m} = W_n B_{m+1} + W_{n-1}(B_m + B_{m-1} - 2B_{m-2}) + W_{n-2}(B_m - 2B_{m-1}) - 2W_{n-3}B_m.$$

Proof. Take $r = 1, s = 1, t = 1, u = -2$ in Theorem 1.8. □

As particular cases of the above theorem, we give identities for $B_{n+m}$ and $C_{n+m}$.

**Corollary 8.4.** *For all integers $m, n$, we have*

$$\begin{aligned} B_{n+m} &= B_n B_{m+1} + B_{n-1}(B_m + B_{m-1} - 2B_{m-2}) + B_{n-2}(B_m - 2B_{m-1}) - 2B_{n-3}B_m, \\ C_{n+m} &= C_n B_{m+1} + C_{n-1}(B_m + B_{m-1} - 2B_{m-2}) + C_{n-2}(B_m - 2B_{m-1}) - 2C_{n-3}B_m. \end{aligned}$$

# 9 CONCLUSIONS

Sequences have been fascinating topic for mathematicians for centuries. The Fibonacci and Lucas sequences are sources of many nice and interesting identities. For example, in [17], authors study on the solutions of the connection problems between Fermat and generalized Fibonacci polynomials. For rich applications of these second order sequences in science and nature, one can see the citations in [18].

As a fourth order sequence, we introduce the generalized Blaise sequence (and it's two special cases, namely, Blaise and Blaise-Lucas sequences) and we present Binet's formulas, generating functions, Simson formulas, the sum formulas, some identities, recurrence properties and matrices for these sequences.

We have shown that there are close relations between Blaise, Blaise-Lucas numbers (which are fourth order linear recurences) and special third order linear recurences (numbers), namely Jacobsthal-Padovan, Jacobsthal-Perrin, adjusted Jacobsthal-Padovan, modified Jacobsthal-Padovan numbers.

Linear recurrence relations (sequences) have many applications. We now present one of them. The ratio of two consecutive Padovan numbers converges to the plastic ratio, $\alpha_P$ (which is given in (9.1) below), which have many applications to such as architecture, see [19]. Padovan numbers is defined by the third-order recurrence relations

$$P_{n+3} = P_{n+1} + P_n, \qquad P_0 = 1, P_1 = 1, P_2 = 1.$$

The characteristic equation associated with Padovan sequence is $x^3 - x - 1 = 0$ with roots $\alpha, \beta$ and $\gamma$ in which

$$\alpha = \left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3} \simeq 1.32471795724 \tag{9.1}$$

is called plastic number (or plastic ratio or plastic constant or silver number) and

$$\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = \alpha.$$

The plastic number is used in art and architecture. Richard Padovan studied on plastic number in Architecture and Mathematics in [20, 21].

Next, we list applications of sequences which are linear recurrence relations.

First, we present some applications of second order sequences.

- For the applications of Gaussian Fibonacci and Gaussian Lucas numbers to Pauli Fibonacci and Pauli Lucas quaternions, see [22].
- For the application of Pell Numbers to the solutions of three-dimensional difference equation systems, see [23].
- For the application of Jacobsthal numbers to special matrices, see [24].
- For the application of generalized k-order Fibonacci numbers to hybrid quaternions, see [25].
- For the applications of Fibonacci and Lucas numbers to Split Complex Bi-Periodic numbers, see [26].
- For the applications of generalized bivariate Fibonacci and Lucas polynomials to matrix polynomials, see [27].
- For the applications of generalized Fibonacci numbers to binomial sums, see [28].
- For the application of generalized Jacobsthal numbers to hyperbolic numbers, see [29].
- For the application of generalized Fibonacci numbers to dual hyperbolic numbers, see [30].
- For the application of Laplace transform and various matrix operations to the characteristic polynomial of the Fibonacci numbers, see [31].
- For the application of Generalized Fibonacci Matrices to Cryptography, see [32].
- For the application of higher order Jacobsthal numbers to quaternions, see [33].
- For the application of Fibonacci and Lucas Identities to Toeplitz-Hessenberg matrices, see [34].
- For the applications of Fibonacci numbers to lacunary statistical convergence, see [35].
- For the applications of Fibonacci numbers to lacunary statistical convergence in intuitionistic fuzzy normed linear spaces, see [36].
- For the applications of Fibonacci numbers to ideal convergence on intuitionistic fuzzy normed linear spaces, see [37].

We now present some other applications of third order sequences.

- For the applications of third order Jacobsthal numbers and Tribonacci numbers to quaternions, see [38] and [39], respectively.
- For the application of Tribonacci numbers to special matrices, see [40].
- For the applications of Padovan numbers and Tribonacci numbers to coding theory, see [41] and [42], respectively.
- For the application of Pell-Padovan numbers to groups, see [43].
- For the application of adjusted Jacobsthal-Padovan numbers to the exact solutions of some difference equations, see [44].
- For the application of Gaussian Tribonacci numbers to various graphs, see [45].
- For the application of third-order Jacobsthal numbers to hyperbolic numbers, see [46].
- For the application of Narayan numbers to finite groups see [47].
- For the application of generalized third-order Jacobsthal sequence to binomial transform, see [48].
- For the application of generalized Generalized Padovan numbers to Binomial Transform, see [49].

- For the application of generalized Tribonacci numbers to Gaussian numbers, see [50].
- For the application of generalized Tribonacci numbers to Sedenions, see [51].
- For the application of Tribonacci and Tribonacci-Lucas numbers to matrices, see [52].
- For the application of generalized Tribonacci numbers to circulant matrix, see [53].

Next, we list some applications of fourth order sequences.

- For the application of Tetranacci and Tetranacci-Lucas numbers to quaternions, see [54].
- For the application of generalized Tetranacci numbers to Gaussian numbers, see [55].
- For the application of Tetranacci and Tetranacci-Lucas numbers to matrices, see [56].
- For the application of generalized Tetranacci numbers to binomial transform, see [57].

We now present some applications of fifth order sequences.

- For the application of Pentanacci numbers to matrices, see [58].
- For the application of generalized Pentanacci numbers to quaternions, see [59].
- For the application of generalized Pentanacci numbers to binomial transform, see [60].

# COMPETING INTERESTS

Author has declared that no competing interests exist.

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