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Research Article

Paracontact Metric (κ, μ)-Manifold Satisfying the Miao-Tam Equation

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In this paper, we classified the paracontact metric (κ, μ)-manifold satisfying the Miao-Tam critical equation with $\kappa > -1$. We proved that it is locally isometric to the product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of negative constant curvature -4.

1. Introduction

Inspired by the positive mass theorem and the variational characterization of Einstein metrics on a closed manifold, with an aim to find a proper concept of metrics that would sit between constant scalar curvature metrics and Einstein metrics, in [1], Miao and Tam studied the variational properties of the volume functional on the space of constant scalar curvature metrics with a prescribed boundary metric. Specifically, they derived the following sufficient and necessary condition for a metric to be a critical point:

Theorem 1 (Theorem 5 in [1]). *Let Ω be a compact n -dimensional Riemannian manifold with smooth boundary Σ, γ be a given metric on Σ, and $\mathscr{M}_\gamma^K$ be the space of metrics on Ω which have constant scalar curvature K and have induced metric on Σ given by γ. Let $g \in \mathscr{M}_\gamma^K$ be a smooth metric such that the first Dirichlet eigenvalue of $(n-1)\Delta_g + K$ is positive. Then, g is a critical point of the volume functional in $\mathscr{M}_\gamma^K$ if and only if there is a smooth function λ on Ω such that $\lambda = 0$ on Σ and*

$$-\left(\Delta_g \lambda\right) g + \nabla_g^2 \lambda - \lambda Ric(g) = g, \quad (1)$$

where Δ_g and ∇_g^2 are the Laplacian and Hessian operators with respect to g, and $Ric(g)$ is the Ricci curvature of g.

For brevity, we call such critical metric as Miao-Tam critical metric and refer to equation (1) as the Miao-Tam equation. A fundamental property of a Miao-Tam critical metric is that its scalar curvature is a constant (see Theorem 7 in [1]). Some explicit examples of Miao-Tam critical metrics can be found in [1, 2], including not only the standard metrics on geodesic balls in space forms but the spatial Schwarzschild metrics and AdS-Schwarzschild metrics restricted to certain domains containing their horizon and bounded by two spherically symmetric spheres. In [2], the authors classified all Einstein and conformally flat Miao-Tam critical metrics. In fact, they proved that any connected, compact, Einstein manifold with smooth boundary satisfying Miao-Tam critical condition is isometric to a geodesic ball in a simply connected space form. And then several generalizations of this rigidity result were found by different authors, replacing the Einstein assumption by a weaker condition such as harmonic Weyl tensor [3], parallel Ricci tensor [4], or cyclic parallel Ricci tensor [5]. For Some other generalizations or rigidity results, we can refer to [6–10], etc.

Recently, some geometricians focus on the study of Miao-Tam equation within the framework of contact metric manifolds. In [11], the authors proved that a complete K-contact metric satisfying the Miao-Tam critical condition is isometric to a unit sphere S^{2n+1}. Furthermore, they studied (k, μ)-contact metrics satisfying the Miao-Tam equation.

Moreover, the Miao-Tam equation within the framework of Kenmotsu and almost Kenmotsu manifolds was studied in [12], and it was proved that a Kenmotsu metric satisfying the Miao-Tam equation is Einstein. In addition, in [13], the authors studied the critical point equation on K-paracontact manifolds; especially, they proved that any K-paracontact manifolds satisfying the Miao-Tam equation must be Einstein. We also note that some geometric structures such as Ricci soliton were studied within the framework of paracontact metric (κ, μ)-manifold (see [14]). In this direction, it is natural to study paracontact metric (κ, μ)-manifold satisfying the Miao-Tam equation. In this paper, we will prove the following main result:

Theorem 2. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a paracontact metric (κ, μ)-manifold of dimensional $(2n+1)$ with $\kappa > -1$. If (g, λ) is a nonconstant solution of the Miao-Tam equation, then M^{2n+1} is locally flat in dimension 3, and in higher dimensions $(n > 1)$, it is locally isometric to the product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of negative constant curvature equal to -4.*

2. Preliminaries

In this section, we recall some basic definitions and facts on paracontact metric manifolds which we will use later. For more details and some examples, we refer to [15–26].

A $(2n+1)$-dimensional smooth manifold M^{2n+1} is said to have an *almost paracontact structure* (φ, ξ, η), if it admits a $(1,1)$-tensor field φ, a vector field ξ, and a 1-form η satisfying the following conditions:

(i) $\eta(\xi) = 1$, $\varphi^2 = \mathrm{i}d - \eta \otimes \xi$

(ii) The tensor field φ induces an almost paracomplex structure on each fiber of $\mathscr{D} = \mathrm{Ker}\,(\eta)$, i.e., the eigendistributions $\mathscr{D}^+$ and $\mathscr{D}^-$ of φ corresponding to the eigenvalues 1 and -1, respectively, have same dimension n

From the definition, it is easy to see that $\varphi\xi = 0$, $\eta \circ \varphi = 0$, and the endomorphism φ have rank $2n$. An almost paracontact structure is said to be *normal* if and only if the tensor field $N_\varphi := [\varphi, \varphi] - 2d\eta \otimes \xi$ vanishes identically. If an almost paracontact manifold admits a pseudo-Riemannian metric g such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2}$$

for all $X, Y \in \Gamma(TM)$, then we say that M has an almost paracontact metric structure, and g is called compatible metric. It follows that $\eta = g(\cdot, \xi)$ and $g(\cdot, \varphi\,\cdot) = -g(\varphi\,\cdot, \cdot)$. Notice that any such a pseudo-Riemannian metric is necessarily of signature $(n+1, n)$.

If in addition $d\eta(X, Y) = g(X, \varphi Y)$ for all vector fields X, Y on M, then the manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ is said to be a *paracontact metric manifold*. In this case, η becomes a contact form, i.e., $\eta \wedge (d\eta)^n \neq 0$, with ξ its Reeb vector field. In a paracontact metric manifold, one defines two self-adjoint operators h and l by $h = 1/2\mathscr{L}_\xi\varphi$ and $l = R(\cdot, \xi)\xi$, where $\mathscr{L}_\xi$ is the Lie derivative along ξ, and R is the curvature tensor of g. It is known in [25] that the two operators h and l satisfy

$$\mathrm{T}rh = 0, h\xi = 0, l\xi = 0, h\varphi = -\varphi h. \tag{3}$$

And there also holds

$$\nabla_X \xi = -\varphi X + \varphi h X, \tag{4}$$

$$\nabla_\xi h = \varphi h^2 - \varphi - \varphi l, \tag{5}$$

where ∇ is the Levi-Civita connection of the pseudo-Riemannian manifold (M, g). Moreover, $h = 0$ if and only if ξ is a Killing vector field, and in this case, the paracontact metric manifold M is said to be a K *-paracontact manifold*. A normal paracontact metric manifold is said to be a *paraSasakian manifold*.

The study of nullity conditions on paracontact geometry is the most interesting topics in paracontact geometry. Motivated by the relationship between contact metric and paracontact geometry, in [18],. Cappelletti Montano et al. introduced the following.

Definition 3. A paracontact metric manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ is said to be a paracontact metric (κ, μ) -manifold, if its curvature tensor R satisfies

$$R(X, Y)\xi = \kappa[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{6}$$

for all tangent vector fields X, Y on M, where κ, μ are real constants.

On a paracontact metric (κ, μ)-manifold $M^{2n+1}(\varphi, \eta, \xi, g)(n \geq 1)$, the following formulas are valid [18]:

$$h^2 = (1+\kappa)\varphi^2, \tag{7}$$

$$Q\xi = 2n\kappa\xi, \tag{8}$$

where Q is the Ricci operator associated with the Ricci tensor Ric.

Paracontact metric (κ, μ)-spaces satisfy (7) but this condition does not give any type of restriction over the value of κ, unlike in contact metric geometry, because the metric of a paracontact metric manifold is not positive definite. However, The geometric behavior of the paracontact metric (κ, μ)-manifold is different according $\kappa < -1$, $\kappa = 1$ and $\kappa > -1$. In particular, for the case $\kappa < -1$ and $\kappa > -1$, (κ, μ)-nullity condition (7) determines the whole curvature tensor field completely. The case $\kappa = -1$ is equivalent to $h^2 = 0$ but not to $h = 0$, which is different from contact (κ, μ)-space. Indeed, there are examples of paracontact metric (κ, μ)-spaces with $h^2 = 0$ but h=0, as was first shown in [18, 27, 28]. In this paper, we consider the paracontact metric (κ, μ)-manifolds with the condition $\kappa > -1$.

3. The Proof of Theorem 2

Before giving the proof of Theorem 2, we introduce some important lemmas which will be used later. First of all, we recall a basic fact about paracontact metric (κ, μ)-manifold.

Lemma 4 (Corollary 4.14 in [18]). *In any $(2n+1)$ -dimensional paracontact metric (κ, μ)-manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ such that $\kappa > -1$, the Ricci operator Q of M is given by*

$$QX = [2(1-n) + n\mu]X + [2(n-1) + \mu]hX + [2(n-1) + n(2\kappa - \mu)]\eta(X)\xi, \tag{9}$$

for any vector field X. In particular, (M, g) is η-Einstein if and only if $\mu = 2(1-n)$, Einstein if and only if $\kappa = \mu = 0$ and $n = 1$ (in this case, the manifold is Ricci-flat). Further, the scalar curvature of M is $2n(2(1-n) + \kappa + n\mu)$.

In the following, we consider paracontact metric (κ, μ) -manifold satisfying the Miao-Tam equation.

Lemma 5. *Let (g, λ) be a nonconstant solution of the Miao-Tam equation on the k-dimensional semi-Riemannian manifold M^k with scalar curvature S. Then, the curvature tensor R can be expressed as*

$$R(X, Y)D\lambda = (X\lambda)QY - (Y\lambda)QX + \lambda(\nabla_X Q)Y - \lambda(\nabla_Y Q)X + (Xf)Y - (Yf)X, \tag{10}$$

for any vector field X, Y on M, where $f = -(\lambda S + 1)/(k-1)$.

Proof. Tracing (1), we obtain

$$\Delta_g \lambda = -\frac{\lambda S + k}{k-1}. \tag{11}$$

Then, the Miao-Tam equation (1) can be exhibited as

$$\nabla_X D\lambda = \lambda QX + fX, \tag{12}$$

for any vector field X on M, where $f = -(\lambda S + 1)/(k-1)$. Taking the covariant derivative of (12) along an arbitrary vector field Y on M, we obtain

$$\nabla_Y(\nabla_X D\lambda) = (Y\lambda)QX + \lambda(\nabla_Y Q)X + \lambda Q(\nabla_Y X) + (Yf)X + f\nabla_Y X. \tag{13}$$

Similarly, we have

$$\nabla_X(\nabla_Y D\lambda) = (X\lambda)QY + \lambda(\nabla_X Q)Y + \lambda Q(\nabla_X Y) + (Xf)Y + f\nabla_X Y, \tag{14}$$

for any vector field X, Y on M. Comparing the preceding two equations and using (12) in the well-known expression of the curvature tensor $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$, we obtain the result.

Lemma 6. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a paracontact metric (κ, μ) -manifold of dimensional $(2n+1)$ with $\kappa > -1$, and (g, λ) be a nonconstant solution of the Miao-Tam equation on M^{2n+1}. Then, we have*

$$\mu(n + \kappa + 1) = 2\kappa. \tag{15}$$

Proof. Firstly, taking covariant derivative of (8) along any vector field X, and using (4), we can obtain

$$(\nabla_X Q)\xi = Q(\varphi X - \varphi hX) - 2n\kappa(\varphi X - \varphi hX). \tag{16}$$

Taking the inner product of (10) with ξ and using (8) and (16), we have

$$\begin{aligned} g(R(X, Y)D\lambda, \xi) = {} & 2n\kappa[(X\lambda)\eta(Y) - (Y\lambda)\eta(X)] \\ & + \lambda g(Q(\varphi X - \varphi hX), Y) \\ & - \lambda g(Q(\varphi Y - \varphi hY), X) + 4\lambda n\kappa g(\varphi Y, X) \\ & + (Xf)\eta(Y) - (Yf)\eta(X), \end{aligned} \tag{17}$$

where $f = -(\lambda S + 1)/(2n)$ (noting that the dimension of M is $2n+1$).

It follows from (6) that $R(\varphi X, \varphi Y)\xi = 0$. Then, replacing X by φX and Y by φY in (17), respectively, we obtain

$$\lambda[Q\varphi + \varphi Q + \varphi Qh + hQ\varphi - 4n\kappa\varphi]X = 0. \tag{18}$$

Since λ is nonconstant on M, it is easy to see that

$$(Q\varphi + \varphi Q + \varphi Qh + hQ\varphi - 4n\kappa\varphi)X = 0. \tag{19}$$

Replacing X by φX in (9), we have

$$Q\varphi X = [2(1-n) + n\mu]\varphi X + [2(n-1) + \mu]h\varphi X. \tag{20}$$

Then, the action of h on the (20) gives

$$hQ\varphi X = [2(1-n) + n\mu]h\varphi X + (1+\kappa)[2(n-1) + \mu]\varphi X, \tag{21}$$

where we have used (7).

Operating (9) by φ, we have

$$\varphi QX = [2(1-n) + n\mu]\varphi X + [2(n-1) + \mu]\varphi hX. \tag{22}$$

Replacing X by hX in (22) and using (7) again, we get

$$\varphi QhX = [2(1-n) + n\mu]\varphi hX + (1+\kappa)[2(n-1) + \mu]\varphi X. \tag{23}$$

Substituting equations (20)-(23) into (19) yields

$$\mu(n + \kappa + 1) = 2\kappa, \tag{24}$$

which completes the proof of Lemma 6.

Next, we will give the complete proof of Theorem 2.

Proof. Firstly, taking $X = \xi$ in (17) gives

$$\begin{aligned} g(R(\xi, Y)\xi, D\lambda) &= g(\kappa[\eta(Y)\xi - Y] - \mu hY, D\lambda) \\ &= \kappa(\xi\lambda)\eta(Y) - \kappa Y\lambda - \mu(hY)\lambda. \end{aligned} \tag{25}$$

Putting $X = \xi$ in (6) and comparing with the forgoing equation, we obtain

$$\kappa D\lambda + \mu hD\lambda - 2n\kappa((\xi\lambda)\xi - D\lambda) - (\kappa(\xi\lambda) + (\xi f))\xi + Df = 0. \tag{26}$$

Noting that the scalar curvature S is a constant, it follows from $f = -(\lambda S + 1)/(2n)$ that

$$2nDf = -SD\lambda. \tag{27}$$

Then, we can obtain from (26) and (27) that

$$\begin{aligned} &2n\kappa D\lambda + 2n\mu hD\lambda - 4n^2\kappa((\xi\lambda)\xi - D\lambda) \\ &\quad - 2n(\kappa(\xi\lambda) + (\xi f))\xi - SD\lambda = 0. \end{aligned} \tag{28}$$

On the one hand, taking $Y = \xi$ in (6), since $h\xi = 0$, it follows that

$$R(X, \xi)\xi = \kappa[X - \eta(X)\xi] + \mu[hX - \eta(X)h\xi] = \kappa\varphi^2 X + \mu hX, \tag{29}$$

which gives

$$l = \kappa\varphi^2 + \mu h. \tag{30}$$

Substituting (7) and (30) in (5), we get

$$\nabla_\xi h = -\mu\varphi h = \mu h\varphi. \tag{31}$$

On the other hand, we obtain from (12) and (8) that

$$\nabla_\xi D\lambda = (2n\kappa\lambda + f)\xi. \tag{32}$$

Next, taking covariant derivative of (28) along ξ and making use of (31) and (32), we have

$$\begin{aligned} &\left(2n\kappa + 4n^2\kappa - S\right)(2n\kappa\lambda + f)\xi + 2n\mu^2 h\varphi D\lambda \\ &\quad - 4n^2\kappa\xi(\xi\lambda)\xi - 2n\kappa\xi(\xi\lambda)\xi - 2n\xi(\xi f)\xi = 0. \end{aligned} \tag{33}$$

Operating this equation by φ shows

$$2n\mu^2 hD\lambda = 0. \tag{34}$$

By the action of h in (34), it follows from (7) that

$$\mu^2(\kappa + 1)\varphi^2 D\lambda = 0. \tag{35}$$

Since we assume that $\kappa > -1$, we divide it into two cases: Case (i): $\mu = 0$; case (ii): $\varphi^2 D\lambda = 0$.

If case (i) occurs, it follows from Lemma 6 that $\kappa = 0$. Hence, the definition of paracontact metric (κ, μ)-manifold gives that $R(X, Y)\xi = 0$ for any vector field X, Y. From Theorem 3.3 of [26], M^{2n+1} is locally flat in dimension 3, and in higher dimensions $(n > 1)$, it is locally isometric to the product of a flat $(n + 1)$-dimensional manifold and an n-dimensional manifold of negative constant curvature -4.

If case (ii) occurs, then $\varphi^2 D\lambda = D\lambda - (\xi\lambda)\xi = 0$, i.e., $D\lambda = (\xi\lambda)\xi$. Differentiating this along an arbitrary vector field X together with (4) implies that

$$\nabla_X D\lambda = X(\xi\lambda)\xi - (\xi\lambda)(\varphi X - \varphi hX). \tag{36}$$

It follows from (12) that $g(\nabla_X D\lambda, Y) = g(\nabla_Y D\lambda, X)$, and then the foregoing equation shows that

$$X(\xi\lambda)\eta(Y) - Y(\xi\lambda)\eta(X) - (\xi\lambda)d\eta(X, Y) = 0. \tag{37}$$

Replacing X by φX, Y by φY, and noting that $d\eta$ is nonzero for any paracontact metric manifolds, it follows that $\xi\lambda = 0$. Hence, $D\lambda = 0$, λ, is a constant, which gives a contradiction.

This completes the proof of Theorem 2.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author(s) declare(s) that they have no conflicts of interest.

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