Stability and Feedback Control of Volterra Type Systems with Time Delay

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

New results for stability and feedback control of time delay systems were proposed. These results were obtained by using Lyapunov Razumikhin method to approximate the stability of the uncontrolled Volterra type system with delay and designing a state feedback controller using a model transformation technique, the Lyapunov matrix equation and the Razumikhin approach for the stabilization of the controlled Volterra type system with delay. Examples are given to illustrate the effectiveness of the theoretical results.

1 Introduction

Stability theory for control systems with time delays is much more complicated and challenging to analyze than for systems without delays. Time-delays in control systems often arise naturally in the system process and

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information distribution to different part of the system; they are frequently observed in models from engineering, biology, economics, as well as other areas of study and has been source of poor system performances and even instability [1,2,3,4]. Studies involving different time delays can be found in ship stabilization, control processes for pressure, and heat transfer regulation, but they are sometimes deliberately introduced into feedback systems to improve system performances see [1,5] and references therein for details. See also [6,7,8], and [9] for more details on system performances in the presence of time delays.

For a given control system with delays in control or state variables, stability is one of the most important characteristics to be determined. The stability of a system implies that small disturbances in the system input (either in system parameters or initial conditions of the system) does not result in considerable changes in the system output. Several methods of analysis are available in studying the stability of such systems which includes the Lyapunov-based [10], fixed point based [11] and spectral radius [12] approaches see [13] and other references therein for details). The Lyapunov based approach which is the focus of stability application in this research is widely used in studying stability theories as well as other qualitative and quantitative properties of linear and nonlinear delay differential systems see [14,15] and references therein. The Lyapunov based approach is classified into two major types; the Razumikhin approach [16,17] and the Krasovskii approach [18]. The Krasovskii's approach often leads to linear matrix inequality results and can be applied to lots of problems that may provide desired conditions, but they are computationally complex and often presents scalability see [19]. The computational difficulty and poor scalability associated with the Krasovskii's approach has prompted the adoption of the Razumikhin approach in this research.

For real life control applications to systems with delays; the desire is to design a system that would be robustly stable and ensure adequate performance. A significant interconnection that can be used to achieve this design is the feedback configuration. The robustness of feedback design on systems response depends on the design goals and methods. Several feedback design goals and formulations of such control problems exists in classical control theory see [20,5,21,22], and [23]. For example, Sipahi et al., [5] studied the stability and stabilization of systems with time delay using eigenvalues, spectrum assignment, parametric techniques, Lyapunov and linear matrix inequality techniques where they discussed problems and opportunities arising from delays in linear time invariant systems modelled by delay differential equations and illustrated that intentional delays, when chosen judiciously can be used to stabilize and improve close-loop response of these systems. In [21], stabilization problem of delay systems was studied under delay-dependent impulsive control where they showed that delays can be introduced into an unstable system to activate stability in the feedback control design strategy using impulsive delay inequality and the Lyapunov method. The robust control design to Furuta system under time delay measurement feedback and exogenous-based perturbation was investigated in [22] where they presented a robust delay-dependent controller based H_∞ theory by using Lyapunov-Krasovskii functional and linear inequalities techniques to design. For the stability analysis of a class of time delay systems Tian et al., [23] have proposed a less conservative stability criterion using the double integral inequality and the Lyapunov-Krasovskii functionals.

The use of the Lyapunov-Razumkhin's approach has received very little attention in the application of stability and feedback control design to control systems of the Volterra type with delay despite its theoretical and practical significance. For example, the study of the control equations of the Volterra type with delays have application in the study of population dynamics and patterns of disease conditions in epidemics and multispecies population interaction in a periodic environment in ecology see [24]. However, the application of the Razumikhin's approach was demonstrated in [24] where they studied the existence and global asymptotic stability of periodic solutions of impulsive Lotka-Volterra type systems and obtained sufficient stability conditions by using a continuation theorem and the Razumikhin's method. Also, various qualitative results to delay integro-differential equations was investigated in [16] by defining suitable Lyapunov function and using the Razumikhin's method to obtain conditions for stability, boundedness, integrable and instability results. The aim of this research is to propose a new stability and feedback control results for time delay systems of the Volterra type by exploring the Lyapunov-Razumikhin's approach. The Razumikhin's approach is adopted in this research because of its ability to yield structurally simpler results with fewer variables and matrix inequalities even though it often leads to a tedious manipulation.

The rest of the paper is organized as follows; Section 2 contains the mathematical notations, preliminaries and definitions. In Section 3, the stability result of the paper is given in terms of Razumikhin type arguments with numerical example. Section 4 contains results on the stabilization of the systems based on the Razumikhins

approach and model transformation technique with examples to illustrate the effectiveness of the proposed results. Finally, Section 5 contains discussions on the simulation output results and the conclusion.

2 Preliminaries and Definitions

Here, we give some preliminaries and Definitions which forms the basis of this study.

2.1 Preliminaries

Suppose $r > 0$ is a given number, $R = (-\infty, \infty), R_{\geq 0} = (0, \infty)$, R^n is a real Euclidean n-space and let $C = C([-r, 0], R^n)$ be the space of continuous function mapping the interval $[-r, R^n]$ into R^n with the norm $\| \cdot \|$, where $\|\phi\| = \sup_{r \leq s \leq 0} |\phi(s)|$. Define $x_t \in C$ by $x_t(s) = x(t + s), -r \leq s \leq 0$, $t \in [0, T)$. If $x(s)$ is a continuous function on $[0, T)$ to R^n then $T > 0$. Here, $T = +\infty$ is allowed.

Consider the time varying delay system

$$\dot{x} = f\,(t, x) \tag{2.1}$$

$$x(t) = \phi(t), t \in [0, T), \text{ where } f(t, x) = A_0 x(t) + A_1 x(t - h) + \int_0^t g(t, s, x(s)) ds\,,$$

and it's control equation

$$\dot{x} = f\,(t, x, u) \tag{2.2}$$

where, $f(t, x, u) = A_0 x(t) + A_1 x(t - h) + \int_0^t g(t, s, x(s)) ds + Bu(t), t > 0$

2.2 Assumptions

Here we make the following assumptions on the system (2.2)

(i). The matrices A_0, A_1, are $n \times n$ constant matrix
(ii). B is an $n \times m$ matrix
(iii). $g(t, s, x(s)): R \times R \times R^n \to R^n$ is continuous and satisfies $|g(t, s, x)| \leq K(t, s)|x|$ with $\int_0^t K(t, s) ds \to 0$ as $t \to \infty$
(iv). The constant delay h is positive with $\dot{h} = 0$

2.3 Definitions

Here, we give some definitions on the subject areas that are required for this research work.

Definition 2.1

The solution $x = 0$ of system (2.1) is stable if given $\varepsilon > 0$ there exist r_0 such that if $|x_0| < r_0$, the $|x(t, x_0)| < \delta$ for all $t \geq 0$.

Definition 2.2

The solution $x = 0$ of system (2.1) is asymptotically stable, if it is stable and there exists a $r_0 > 0$ such that if $|x_0| < r_0$ then $|x(t, x_0)| \to 0$ as $t \to \infty$.

3 Razumikhin Approach for Stability

Here, we give stability results for system (2.1) using Razumikhin type argument for Volterra type system with time delays. For the time delay system (2.1) it is necessary to use the Lyapunov functions of the form;

$$\dot{V}(t,\phi) = \lim \sup_{r\to 0^+} \frac{1}{r}[V(t+r, x(t,\phi)(x+r) - V(t,\phi)],$$

where $V: R \times R^n \to R$ is a continuous function and $\dot{V}(t,\phi)$ is the derivative of V along the solutions of equation (2.1) and $x(t,\phi)$ is the solution of equation (2.1) through (t,ϕ). The proofs of the next two theorems follows the form of [25] and [26].

Theorem 3.1

Let $f: R \times C \to R^n$ be a continuous function that maps $R \times$ (bounded sets of C) into bounded sets of R^n, suppose there is a continuous function $l(s)$ for $s \geq 0$ such that $l(s) > s$ and continuous, non-decreasing function, $u(s), v(s), w(s)$ with, $u(s), v(s), w(s) > 0$ for $s > 0$ and $u(0), v(0), w(0) = 0$. Let $x(t)$ be a solution of system (2.1) on $[0,T], T \leq \infty$, If there is a continuous function $V: R \times R^n \to R$ such that

(i) $u(|x|) \leq V(t,x) \leq v(|x|), t \in R, x \in R^n$
(ii) $V(s, x(s)) < l\left(V(t, x(t))\right)$ for $s \in R_{\geq 0}, t > 0$, where $t_0 = \max \ [0, t-r]$
(iii) $\dot{V}(t, x(t)) \leq -w(|x(t)|)$

then the zero solution of the system (2.1) is asymptotically stable. Here $w(s)$ and r may depend on the solution $x(t)$ as well as V and l.

Proof: Let $x(t)$ be a solution of system (2.1) bounded on $[0,\infty)$, we define $M = \min_{t\geq 0}|x(t)|$ and let $\varepsilon > 0$ be given so that $u(\varepsilon) < v(\varepsilon)$. Then there exists a number $a = a(\varepsilon) > 0$, such that $l(s) - s > a, s \in [u(\varepsilon), v(M)]$. Let $N = N(\varepsilon) > 0$ be the smallest integer such that $v(M) \leq u(\varepsilon) + Na$, and define $\varepsilon_j = u(\varepsilon) + (N - j)a$, $j = 0, 1, 2, \dots, N$. We observe that $V(t, x(t)) \leq \varepsilon_0$ for $t \geq 0$. Suppose $V(t, x(t)) \geq \varepsilon_1$ for all $t \geq r$, then $V(|x(t)|) \geq \varepsilon_1$ for any such t, and hence $|x(t)| \geq v^{-1}(\varepsilon_1) > 0$. Also, for such t, $u(\varepsilon) \leq V(t, x(t)) \leq v(M)$, so that $l\left(V(t, x(t))\right) > V(t, x(t)) + a \geq u(\varepsilon) + (N - 1)a + a = u(\varepsilon) + Na$. But $V(s, x(s)) \leq u(\varepsilon) + Na$ for all $s \geq 0$ and thus for $s \in [t - r, t], t \geq r$. Using condition (iii) with $j = 0$, we get

$$\dot{V}(t, x(t)) \leq -w(|x(t)|), \qquad t \geq r \tag{3.1}$$

Define, $P_1 = v^{-1}(\varepsilon_1)$ and $\eta_1 = \inf_{s\in[P_1, M]} w(s) > 0$, then from condition (ii) we have that $V(t, x(t)) \leq V(r, x(r)) - \eta_1(t - r) \leq \varepsilon_0 - \eta_1(t - r)$ for all $t \geq r$. Since $V(t, x(t))$ is nonnegative, it is a contradiction. So there exists a $t_1 > r$ such that $V\left(t_1', x(t_1')\right) < \varepsilon_1$. If $V(t_1, x(t_1)) = \varepsilon_1$ for some $t_1' > t_1$ we assume that t_1' is chosen such that $V(t, x(t)) < \varepsilon_1$ for $t \in [t_1, t_1')$ and it follows clearly that

$$\dot{V}\left(t_1', x(t_1')\right) \geq 0 \tag{3.2}$$

But, $l(\varepsilon_1) = l\left(V\left(t_1', x(t_1')\right)\right) > V\left(t_1', x(t_1')\right) + a = \varepsilon_1 + a = \varepsilon_0$ since $V(s, x(s)) \leq \varepsilon_0$ for $s \in [t_1' - r, \ t_1']$, it follows again from condition (iii) that $\dot{V}\left(t_1', x(t_1')\right) \leq -w(|x(t_1')|) < 0$, which is a contradiction to (3.2). It follows then that

$$V(t, x(t)) < \varepsilon_1 \tag{3.3}$$

for all $t \geq t_1$. Suppose $V(t, x(t)) \geq \varepsilon_2$ for all $t \geq t_1$, then for $t \geq t_1 + r$, we have $v(|x(t)|) \geq \varepsilon_2$ and therefore $|x(t)| \geq v^{-1}(\varepsilon_2)$. Now, define $P_2 = v^{-1}(\varepsilon_2)$ so that $\varepsilon_2 \leq V(t, x(t)) \leq \varepsilon_1$ for $t \geq t_1 + r$, it follows then that for such for t; $l\left(V(t, x(t))\right) > V(t, x(t)) + a \geq u(\varepsilon) + (N - 1)a = \varepsilon_1 \geq V(s, x(s))$ for some $s \in [t - r, t]$. Then, by condition (iii), we have that

$$\dot{V}(t, x(t)) \leq -w(|x(t)|), t \geq t_1 + r \tag{3.4}$$

If $\eta_2 = \inf_{s \in [P_2, M]} w(s)$, then $\eta_2 > 0$ and from (3.4) we have $V\big(t, x(t)\big) \le V\big(t_1 + r, x(t_1 + r)\big) - \eta_2(t - t_1 - r) \le \varepsilon_1 - \eta_2(t - t_1 - r)$. But, for very large $t \ge t_1 + r$ this leads to a contradiction. So there exists $t_2 \ge t_1 + r$ such that $V\big(t_2, x(t_2)\big) \le \varepsilon_2$, suppose for some $t_2' > t_2$, $V\left(t_2', x(t_2')\right) = \varepsilon_2$, while $V\big(t, x(t)\big) < \varepsilon_2$ for $t \in [t_2, t_2')$. It follows that

$$\dot{V}\left(t_1', x(t_1')\right) \ge 0 \tag{3.5}$$

However, $l(\varepsilon_2) = l\left(V\left(t_2', x(t_2')\right)\right) > V\left(t_2', x(t_2')\right) + a = \varepsilon_2 + a = \varepsilon_1$. But also $V\big(s, x(s)\big) \le \varepsilon_1$ for $s \in [t_2' - r, \, t_2']$, this follows from (3.3) since for each $s, s \ge t_2 - r \ge t_1$. So $l\left(V\left(t_2', x(t_2')\right)\right) > V\big(s, x(s)\big)$ for $s \in [t_2' - r, \, t_2']$ and using condition (iii), we get $\dot{V}\left(t_2', x(t_2')\right) < 0$ which contradicts (3.5). So there exists $V\big(t, x(t)\big) \le \varepsilon_2$ for $t \ge t_2$, continuing in this way, we get $j = 0, 1, \cdots, N$, that is there exists t_j such that $V\big(t, x(t)\big) \le \varepsilon_j$ for $t \ge t_2$, where $t_j \ge t_{j-1} + r$, and $t_0 = 0$. But $\varepsilon_N = u(\varepsilon)$; that is, $V\big(t, x(t)\big) < u(\varepsilon)$ for $t \ge t_N$. Thus, for such t, we have $u(|x(t)|) < u(\varepsilon)$ from which we get $\|x(t)\| < \varepsilon$ for $t \ge t_N$ and the proof is complete.

Lemma 3.1

Suppose all of the conditions of Theorem 3.1 are satisfied and $x = 0$ is stable for the system (2.1), then it is uniformly asymptotically stable.

Proof: The proof follows immediately from the theorem; since the system (2.1) is stable at $x = 0$, there exists a $r_0 > 0$ such that the solution $\|x_t(t, \phi)\| \le l$ for $|\phi| \le r_0$,

Lemma 3.2

Suppose all of the conditions of Theorem 3.1 are satisfied for any solution not necessarily bounded on $[0, \infty)$, then $x = 0$ is asymptotically stable for system (2.1).

Proof: If conditions (ii) and (iii) holds for any solution of system (2.1), then the condition implies $\dot{V}\big(t, x(t)\big) \le 0$ for any solution $x_t(t, x_0)$ of system (2.1) for which $x_s(s, \phi)$ is R^n and $l\left(V\big(t, x(t)\big)\right) > V\big(s, x(s)\big)$ for $0 \le s \le l$ and therefore $\dot{V}(t, \phi) = \lim \sup_{h \to 0^+} \frac{1}{h}[V(t + h, x(t, \phi)(x + h) - V(t, \phi)]$, which implies that the point $x = 0$ is stable for system (2.1) and the proof is complete.

3.1 Main result on stability

The results of the theorem and lemmas will now be used to investigate the asymptotic stability of the system (2.1).

Theorem 3.2

Let all the assumptions of equation (3.1) be satisfied and suppose that

a). $\|P\| \left(\|A_2\| + \int_0^t K(t, s) ds\right) < \frac{\lambda}{2\sigma}$ \hfill (3.6)

b). Given $\varepsilon > 0, M > 0$ there exists a $k > 0$ such that $\frac{\partial V}{\partial x} \cdot \int_0^t g\big(t, s, x(s)\big) ds \le \varepsilon$ fo $t \ge kr$, $|x| \le M$ and if $x(t)$ is a solution of system (2.1) satisfying condition (ii) of Theorem 3.1 for $t - kr \le s \le t, t \ge kr$. Then

$\frac{\partial V}{\partial x} \cdot \left[A_0 x(t) + A_1 x(t - h) + \int_0^t g(t, s, x(s)) ds\right] + \frac{\partial V}{\partial t} \le -w|x(t)|$ for $t \ge kr$. If $x(t)$ is a bounded solution of system (2.1), $x(t) \to 0$ as $t \to +\infty$ where, $\frac{\partial V}{\partial x} = \left(\frac{\partial V}{\partial x_1} \cdots \cdots \frac{\partial V}{\partial x_n}\right)$.

then the zero solution of equation (2.1) is asymptotically stable.

Proof: To show that condition (a) of Theorem 3.2 is satisfied. Let there exist a function $V: R \times R^n \to R$ having continuous first partial derivatives in all variables. It is known from matrix theory [27] that there exists a symmetric positive definite matrix P such that the Lyapunov matrix equation $PA + A^T P + I = 0$, where I is the identity matrix and A^T is the transpose of A. Let λ and α be positive numbers such that λ^2 and α^2 are the least and greatest eigen-values of P respectively. Define $V(x) = < P\,x, x >$, it is clear then that;

$$\lambda^2 x^2 \leq < P\,x, x > \leq \alpha^2 x^2, \forall x \in R^n.$$

Given inequality (3.6), choose $\mu > 1$ so that $1 - \frac{2\mu\sigma\|P\|\left(\|A_2\| + \int_0^t K(t,s)ds\right)}{\lambda} = \rho > 0$, for any $\tau \in (\tau_1, \infty)$, we consider the system

$$\dot{x}(t) = A_1 x(t) + A_2 x(t - h) + \int_{t-\tau}^t g\big(t, s, x(s)\big) ds, t \geq \tau \tag{3.7}$$

Set $V(x) = < P\,x, x >$, we shall prove that the function $V(\phi)$ satisfies all the conditions of the Razumikhin theorem, that is Theorem 3.1 for system (2.1). It is obvious that, the conditions (ii) and (iii) of the Theorem 3.1 holds. Assume now that $\mu^2 V(\phi) \geq V(\phi(\theta))$ so that $\mu^2 \alpha^2 |\phi|^2 \geq \lambda^2 |\phi(\theta)|^2$ and hence $|\phi(\theta)| \leq \frac{\mu\alpha|\phi(\theta)|}{\lambda}$ for all $\theta \in (t - \tau, \infty)$, then, the derivative $V(t, \phi)$ of V along the solution of equation (2.1) is given by

$$\dot{V}(t, \phi) = \left\langle P \left[A_1 \phi(0) + A_2 \phi(-h) + \int_{t-\tau}^t g\big(t, s, x(s)\big) ds \right], \phi \right\rangle$$

$$+ \left\langle P\phi, A_1 \phi(0) + A_2(-h) + \int_{t-\tau}^t g\big(t, s, x(s)\big) ds \right\rangle$$

$$= \left\langle (PA_1 + A^T, P)\phi(0), \phi \right\rangle + 2 \left\langle P\phi, A_2 \phi(-h) + \int_{t-\tau}^t g\big(t, s, x(s)\big) ds \right\rangle$$

$$\leq -|\phi|^2 + 2|\phi|\|P\| \left(\|A_2\| + \int_{t-\tau}^t g\big(t, s, x(s)\big) ds \right) \cdot \sup_{t-\tau \leq \theta \leq t} |\phi(\theta)|$$

$$\leq -\rho|\phi|^2$$

This implies the conditions of the Razumikhin theorem given in Theorem 3.1 are satisfied as $w(s) = \rho s^2$ and $l(s) = \mu^2 s$. Therefore the zero solution (2.1) is asymptotically stable with r and w depending on x, V and l. Furthermore, we show that condition (b) also holds, let $l(s) = \mu^2 s$, then for any positive integer k and any solution $x(t)$ of (2.1) we have $\left| \int_{t-k\tau}^t g\big(t, s, x(s)\big) ds \right| \leq \int_{t-k\tau}^t K(t,s)|x(s)|ds \leq \sup_{t-k\tau \leq \theta \leq t} |x(\theta)| \int_0^t K(t,s)ds$, $t \geq k\tau$, and $x(t)$ is a solution of (2.1) satisfying $\langle P\,x, x \rangle < \mu^2 Px^2$. It follows then that for such k, t and x; $\left| 2Px \cdot \int_{t-k\tau}^t g\big(t, s, x(s)\big) ds \right| \leq \frac{2\mu\sigma|P||x|^2}{\lambda}$, and condition (b) is satisfied by the definition of ρ and w. Furthermore, the inequality $\left| \int_0^{t-k\tau} g\big(t, s, x(s)\big) ds \right| \leq |x(s)| \int_0^t K(t,s)ds \to 0$ as $t \to \infty$. Hence, by the assumptions on system (2.1) every bounded solution of (2.1) tends to zero as $t \to \infty$ and the proof is complete.

3.2 Numerical example on stability of the system

Here, we give numerical example to illustrate the use of the Razumikhin theory as an application to Theorem 3.2.

Example 3.1

Consider the delay system (2.1) with $A_1 = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}, A_2 = \begin{pmatrix} -1/4 & 0 \\ -1/4 & -1/4 \end{pmatrix}$ and $g(t,s,x(s)) = \begin{pmatrix} 0 \\ \cos(t-3) \cdot x(t) \end{pmatrix}$. Now, let $P = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.5 \end{pmatrix}$ be a symmetric positive definite matrix with $\lambda^2 = 0.0625$ and $\alpha^2 = 0.2500$ as the least and greatest eigen-values, and observe that,

$$\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0.25 & 0 \\ 0 & 0.5 \end{pmatrix} + \begin{pmatrix} 0.25 & 0 \\ 0 & 0.5 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix},$$

satisfies the Lyapunov matrix equation. Also check that $\|P\| \left(\|A_2\| + \int_0^t K(t,s)ds \right) < \frac{\lambda}{2\sigma}$ by inequality (3.6) is satisfied with $\|P\| \left(\|A_2\| + \int_0^t K(t,s)ds \right) = 0.2023$ and $\frac{\lambda}{2\sigma} = 0.25$ and $\rho = 1 - \frac{2\mu\sigma\|P\|\left(\|A_2\|+\int_0^t K(t,s)ds\right)}{\lambda} = 1 - \frac{2(1.1)\times0.5\times0.4045}{0.25} = 1 - 0.8899 = 0.11 > 0$ and all the conditions of Theorem 3.2 are satisfied with $K(t,s) = \cos(t-3)$ and $\int_0^t K(t,s)ds \to 0$ as $t \to \infty$, $\mu = 1.1$. Therefore system (2.1) is asymptotically stable.

Example 3.2

Consider the delay system (2.1) with $A_1 = \begin{pmatrix} -2 & 1 \\ 2 & -4 \end{pmatrix}, A_2 = \begin{pmatrix} -0.1 & 0 \\ 0.1 & -0.2 \end{pmatrix}$ and $g(t,s,x(s)) = \begin{pmatrix} 0 \\ (e^{-x}\sin x - e^{-2x}) \cdot x(t) \end{pmatrix}$. Now, let $P = \begin{pmatrix} 0.3194 & 0.1389 \\ 0.1389 & 0.1944 \end{pmatrix}$ be a symmetric positive definite matrix with $\lambda^2 = 0.0109$ and $\alpha^2 = 0.1675$ as the least and greatest eigen-values, and observe also that $\|P\| \left(\|A_2\| + \int_0^t K(t,s)ds \right) < \frac{\lambda}{2\sigma}$ by inequality (3.6) is satisfied with $\|P\| \left(\|A_2\| + \int_0^t K(t,s)ds \right) = 0.0936$ and $\frac{\lambda}{2\sigma} = 0.1278$ and $\rho = 1 - \frac{2\mu\sigma\|P\|\left(\|A_2\|+\int_0^t K(t,s)ds\right)}{\lambda} = 1 - \frac{2(1.1)\times0.4092\times0.0936}{0.1049} = 1 - 0.8033 = 0.1967 > 0$ and all the conditions of Theorem 3.2 are satisfied with $K(t,s) = e^{-x}\sin x - e^{-2x}$ and $\int_0^t K(t,s)ds \to 0$ as $t \to \infty$, $\mu = 1.1$. Therefore, system (2.1) is asymptotically stable.

4 Feedback Stabilization of the System

Here, we use the model transformation technique in [13] to analyze the stabilization of the system (2.2) as follows. Let there exists a $k > 0$ such that for $t \geq k\tau$

$$\left| \int_0^t g(t,s,x(s))ds \right| \leq \left| \int_0^{t-k\tau} g(t,s,x(s))ds + \int_{t-k\tau}^t g(t,s,x(s))ds \right|,$$

where,

$$\left| \int_0^{t-k\tau} g(t,s,x(s))ds \right| \leq |x(s)| \int_0^{t-kr} K(t,s)ds \leq \mu_1|x(t)|;$$

$$\left| \int_{t-kr}^t g(t,s,x(s))ds \right| \leq \sup_{\theta \in [t-k,t]}|x(\theta)| \int_{t-kr}^t K(t,s)ds \leq \mu_2|x(t)|, \text{ and}$$

$$x(t) - x(t-h) = \int_{-h}^0 \dot{x}(t+\theta)d\theta, t \geq h \tag{4.1}$$

Applying this to (2.2)

$$\dot{x}(t) = (A_1 + A_2)x(t) - A_2 \int_{-h}^0 \dot{x}(t+\theta)d\theta + \int_0^t g(t,s,x(s))ds + Bu \tag{4.2}$$

for $x(t) = \phi(t), t \in [-2h, 0], h > 0$

Define, $\vartheta = \frac{\lambda_{min}(Q)}{2\lambda_{max}(P)}$, $\delta = \sqrt{\frac{\lambda_{min}(P)}{\lambda_{max}(P)}}$

Let P and Q be symmetric positive definite matrices in the Lyapunov equation below

$$(A_1 + A_2)^T P + P(A_1 + A_2) = -Q, \tag{4.3}$$

where $A_1 + A_2$ is Hurwitz stable. We associate (4.2) with a state feedback controller $u(t)$ of the form

$$u(t) = -B^T P x(t), \tag{4.4}$$

where $P \in R^{nxn}$ is a symmetric positive definite matrix to be designated; the closed-loop design for equation (2.2) using equation (4.3), (4.4) and the transformed equation (4.2) is given by;

$$\dot{x}(t) = ((A_1 + A_2) - BB^T P)x(t) - A_2 \int_{-h}^{0} \dot{x}(t+\theta)d\theta + \int_{0}^{t} g(t,s,x(s))ds. \tag{4.5}$$

We now ensure that the system (4.5) is asymptotically stable and the closed-loop system is stabilized.

4.1 Designing a guaranteed controller

Here, we use the Lyapunov matrix equation and the Razumikhin approach to stabilize the closed-loop system (4.5).

Theorem 3.3

Suppose $A_1 + A_2$ is asymptotically stable and there exists positive-definite matrices P and Q satisfying equation (4.3) and σ, λ are as defined in the proof of Theorem 3.2 then, the system (4.5) is asymptotically stable if

$$\frac{\vartheta - \|BB^T P\| - \mu_1 \sigma/\lambda - \mu_2 \sigma/\lambda}{\delta[\|A_2\|(\|A_1 + BB^T P\| + \|A_2\| + \mu_1 + \mu_2)]} > 0 \tag{4.6}$$

Proof: consider equation (4.3) given by $(A_1 + A_2)^T P + P(A_1 + A_2) = -Q$ and take the following positive-definite function as the Lyapunov function:

$$V(x_t) = x^T(t)P\,x(t) \tag{4.7}$$

Now, taking the derivative of V in (4.7) along the solution of (4.5) gives

$$\dot{V}(x) = \left[(A_1 + A_2 - BB^T P)x(t) - A_2 \int_{-h}^{0} \dot{x}(t+\theta)d\theta + \int_{0}^{t} g(t,s,x(s))ds \right] \times Px(t)$$

$$+ x(t)^T P \left[(A_1 + A_2 - BB^T P)x(t) - A_2 \int_{-h}^{0} \dot{x}(t+\theta)d\theta + \int_{0}^{t} g(t,s,x(s))ds \right]$$

$$= 2x^T(t)P \left[(A_1 + A_2 - BB^T P)x(t) - A_2 \int_{-h}^{0} \dot{x}(t+\theta)d\theta + \int_{0}^{t} g(t,s,x(s))ds \right]$$

$$= x^T(t)[(A_1 + A_2)^T P + P(A_1 + A_2) - 2PBB^T P]x(t) - 2x^T(t)PA_2 \int_{-h}^{0} \dot{x}(t+\theta)\,d\theta$$

$$+ 2x^T(t)P \int_{0}^{t} g(t,s,x(s))ds \tag{4.8}$$

We can further estimate the following expressions in (4.8) as follows

$$2\,x^T(t)PA_2\int_{-h}^{0}\dot{x}(t+\theta)\,d\theta$$

$$\leq 2\,x^T(t)PA_2\int_{-h}^{0}[(A_1-BB^TP)\times x(t+\theta)+A_2x(t-h+\theta)+\mu_1x(t+\theta)$$
$$+\mu_2x(t-kr+\theta)]d\theta$$

$$2\,x^T(t)P\int_{0}^{t}g(t,s,x(s))ds\leq 2\,x^T(t)P\int_{0}^{t-kr}g(t,s,x(s))ds+2\,x^T(t)P\int_{t-kr}^{t}g(t,s,x(s))ds$$

Using condition (b) of Theorem 3.2 we get

$$2\,x^T(t)P\int_{0}^{t-kr}g(t,s,x(s))ds\leq 2\,x^T(t)P\int_{0}^{t-kr}K(t,s)ds\cdot x(t)\leq\frac{2\,x^T(t)P\sigma\mu_1x(t)}{\lambda}$$

$$2\,x^T(t)P\int_{t-kr}^{t}g(t,s,x(s))ds\leq 2\,x^T(t)P\int_{t-kr}^{t}K(t,s)ds\cdot x(t)\leq\frac{2\,x^T(t)P\sigma\mu_2x(t)}{\lambda}$$

Substituting these estimates into the state feed-back controller in equation (4.5) we get the overall derivative of V along the solution of (4.5) as

$$\dot{V}(x_t)\leq x^T(t)[(A_1+A_2)^TP+P(A_1+A_2)-2\,PBB^T]x(t)$$
$$-2\,x^T(t)PA_2\int_{-h}^{0}[(A_1-BB^TP)\times x(t+\theta)+A_2x(t-h+\theta)+\mu_1x(t+\theta)$$
$$+\mu_2x(t-kr+\theta)]d\theta+\frac{2\,x^T(t)P\sigma\mu_1x(t)}{\lambda}+\frac{2\,x^T(t)P\sigma\mu_2x(t)}{\lambda}$$

$$\leq x^T(t)[(A_1+A_2)^T+P(A_1+A_2)]x(t)-2x^T(t)PBB^TPx(t)$$
$$-2\,x^T(t)PA_2\int_{-h}^{0}[(A_1-BB^TP)\times x(t+\theta)+A_2x(t-h+\theta)+\mu_1x(t+\theta)$$
$$+\mu_2x(t-kr+\theta)]d\theta+\frac{2\,x^T(t)P\sigma\mu_1x(t)}{\lambda}+\frac{2\,x^T(t)P\sigma\mu_2x(t)}{\lambda} \tag{4.9}$$

Now, using the Razumikhin theorem, assume $q>1$ for any non-negative number, the following holds:

$$V(x(\xi))<q^2V(x(t)),t-2h<\xi\leq t. \tag{4.10}$$

Hence,

$$x\|x(\xi)\|<q\delta\|x(t)\|. \tag{4.11}$$

Substituting equation (4.11) into (4.9) gives the following inequality

$$\dot{V}(x_t)=-w\|x(t)\|^2, \tag{4.12}$$

where,

$$w=\lambda_{min}(Q)-2\{\|BB^TP\|+\sigma\mu_1/\lambda+\sigma\mu_2/\lambda+q\delta h[\|A_2\|(\|A_1+BB^TP\|+\|A_2\|+\mu_1+\mu_2)]\}\lambda_{max}(P)$$

now, the derivative (4.10), (4.11) and (4.12) by the Razumikhin theory implies that $\dot{V}(x_t)<0, w>0$ based on the proof of the above Theorem. Thus, by the Razumikhin Theorem it is asymptotically stable.

Remark 4.1

The choice of equation (4.3) guarantees $Q>0$, with $P=I$ and maximizes δ when $P=I$. The maximum bound for the time delay becomes

$$h^* = \vartheta + \|BB^TP\| + \|A_2\|(\|A_1 + BB^TP\| + \|A_2\| + \mu_1 + \mu_2), \tag{4.13}$$

for $0 \leq h \leq h^*$.

4.2 Examples on feedback stabilization of the system

The aim here is to give numerical examples as an illustration to the methods proposed.

Example 4.1

Consider the system

$$\dot{x}(t) = A_1x(t) + A_2x(t - h) + Bu(t) + \int_0^t g(t, s, x(s))ds. \tag{4.14}$$

Here, A_1, A_2 and g are as defined in Example 3.1 of Section 3.2; where $A_1 + A_2 = \begin{pmatrix} -2.25 & 0 \\ -0.25 & -1.25 \end{pmatrix}$ is asymptotically stable, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $P = \begin{pmatrix} 0.2222 & -0.0159 \\ -0.0159 & 0.4032 \end{pmatrix}$ is a symmetric positive definite matrix with $\lambda^2 = 0.0488$ and $\alpha^2 = 0.1637$ as the least and greatest eigen-values, and observe that,

$$\begin{pmatrix} -2.25 & -0.25 \\ 0 & -1.25 \end{pmatrix}\begin{pmatrix} 0.2222 & -0.0159 \\ -0.0159 & 0.4032 \end{pmatrix} + \begin{pmatrix} 0.2222 & -0.0159 \\ -0.0159 & 0.4032 \end{pmatrix}\begin{pmatrix} -2.25 & 0 \\ -0.25 & -1.25 \end{pmatrix},$$

satisfies the Lyapunov matrix equation. Now, set $Q = I$ and observe that,

$$\lambda_{min}(P) = 0.2208 \,, \lambda_{max}(P) = 0.4046, \lambda_{min}(Q) = 1, \vartheta = 1.2359 \,, \delta = 0.7387$$

$\|A_2\|(\|A_1 + BB^TP\| + \|A_2\| + \mu_1 + \mu_2) = 1.1359 \,, \vartheta - \|BB^TP\| = 0.8324 \,, \int_0^t K(t,s)ds \to 0$ as $t \to \infty$. Using inequality (4.6) gives $h = 0.9920$ with a maximum bound $0 \leq h < h^* = 2.3718$. The stabilizing control law $u(t)$ when $Q = I$,

$$u(t) = -BB^TPx(t) = -[0 \ 1]\begin{bmatrix} 0.2222 & -0.0159 \\ -0.0159 & 0.4032 \end{bmatrix} = [0.0159 - 0.4032] \, x(t).$$

Example 4.2

Now, setting $P = I$, using equation (4.3) gives $(Q) = \begin{pmatrix} 4.50 & 0.25 \\ 0.25 & 2.50 \end{pmatrix}$ with $\lambda_{min}(P) = 1$, $\lambda_{max}(P) = 1$, $\delta = 1, (Q) = \begin{pmatrix} 4.50 & 0.25 \\ 0.25 & 2.50 \end{pmatrix}$, $\vartheta = 0.2346$ which gives $h = 0.1704$ and $h^* = 3.6117$

Example 4.3

Let A_1, A_2, and g be defined as in Example 3.2 of Section 3.2, where

$A_1 + A_2 = \begin{pmatrix} -2.1 & 1 \\ 2.1 & -4.2 \end{pmatrix}$ is asymptotically stable, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $P = \begin{pmatrix} 0.2995 & 0.1290 \\ 0.1290 & 0.1835 \end{pmatrix}$ is a symmetric positive definite matrix with $\lambda^2 = 0.0100$ and $\alpha^2 = 0.1466$ as the least and greatest eigen-values respectively. Now, set $Q = I$ and observe that, $\lambda_{min}(P) = 0.1001 \,, \lambda_{max}(P) = 0.3829 \,, \lambda_{min}(Q) = 1 \,, \vartheta = 1.3058 \,, \delta = 0.5113 \,, \vartheta - \|BB^TP\| = 1.0815$

$\|A_2\|(\|A_1 + BB^TP\| + \|A_2\| + \mu_1 + \mu_2) = 1.1433 \,, \int_0^t K(t,s)ds \to 0$ as $t \to \infty$. Using inequality (4.6) gives $h = 1.8500$ with a maximum bound $0 \leq h < h^* = 2.6734$. The stabilizing control law $u(t)$ when $Q = I$,

$$u(t) = -BB^TPx(t) = -[0 \ 1]\begin{bmatrix} 0.2995 & 0.1290 \\ 0.1290 & 0.1835 \end{bmatrix} = [-0.1290 - 0.1835] \, x(t).$$

Example 4.4

Now, setting $P = I$, using equation (4.3) gives $Q = \begin{pmatrix} 4.2 & -3.1 \\ -3.1 & 8.4 \end{pmatrix}$ with $\lambda_{min}(P) = 1, \lambda_{max}(P) = 1, \delta = 1,$ $\vartheta = 1.2779$ which gives $h = 0.7415$ and $h^* = 2.6527$.

5 Discussion and Conclusion

The MATLAB simulation outputs for different values of delay for both controlled and uncontrolled systems of some examples in Section 4.4 are given below.
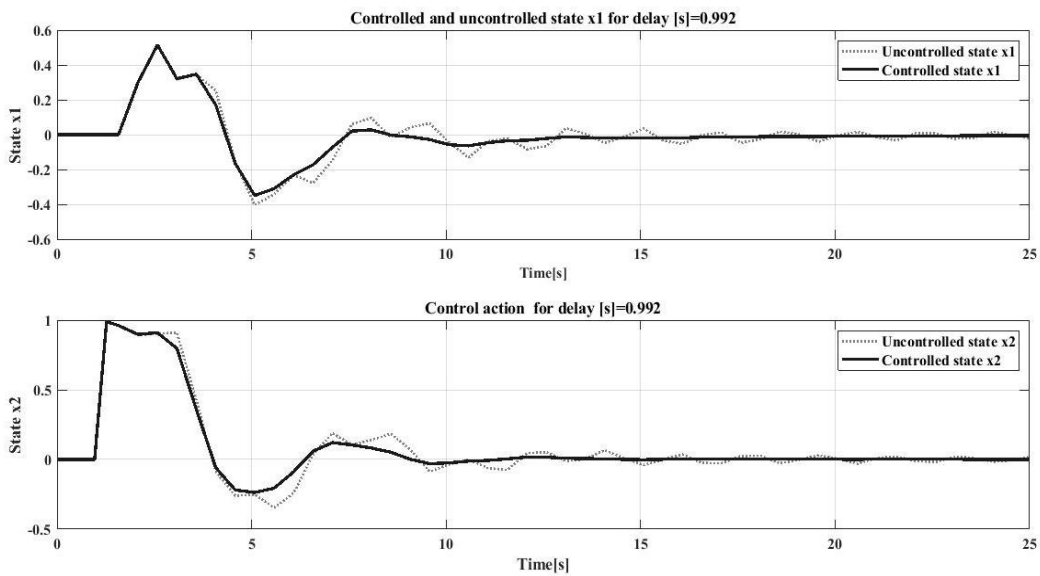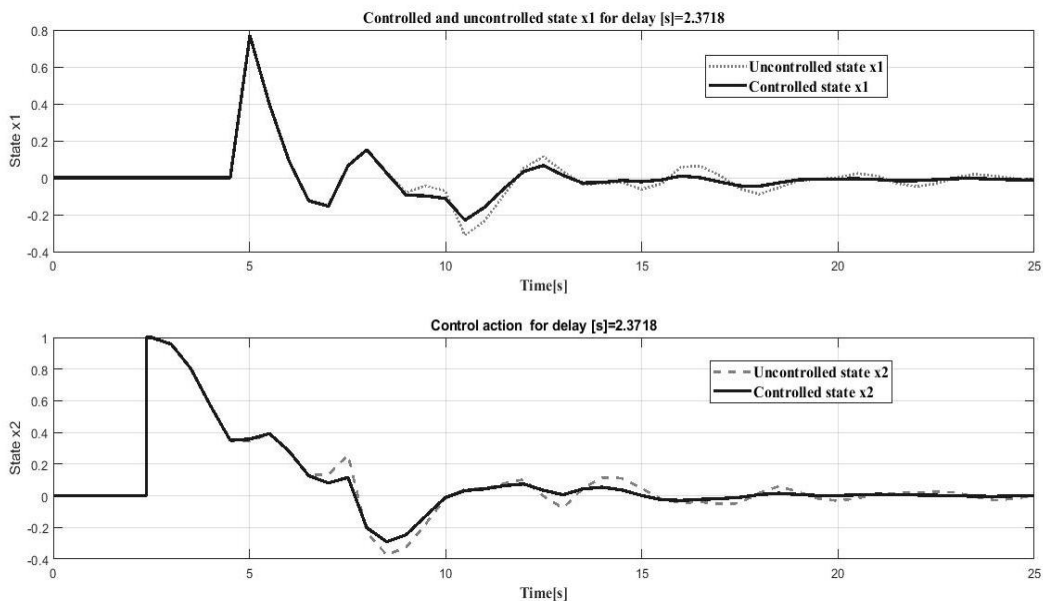

Fig. 1. Controlled and uncontrolled states with delay $h = 0.9920$

Fig. 2. Controlled and uncontrolled states with delay $h = 2.3718$
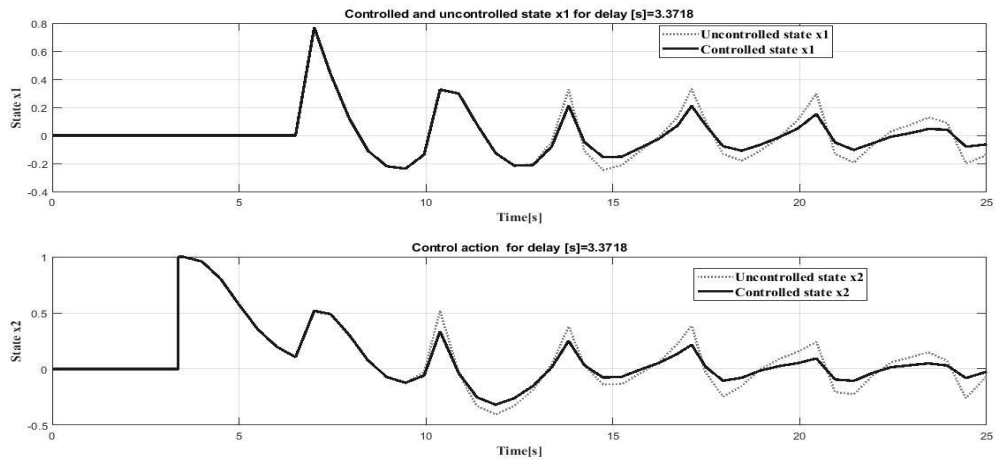
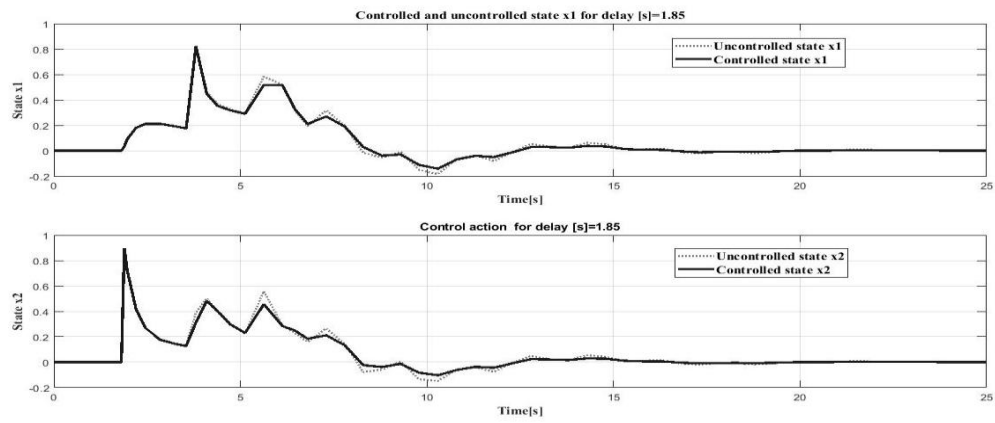**Fig. 3. Controlled and uncontrolled states with delay $h = 3.3718$**

Fig. 4. Controlled and uncontrolled states with delay $h = 1.8500$
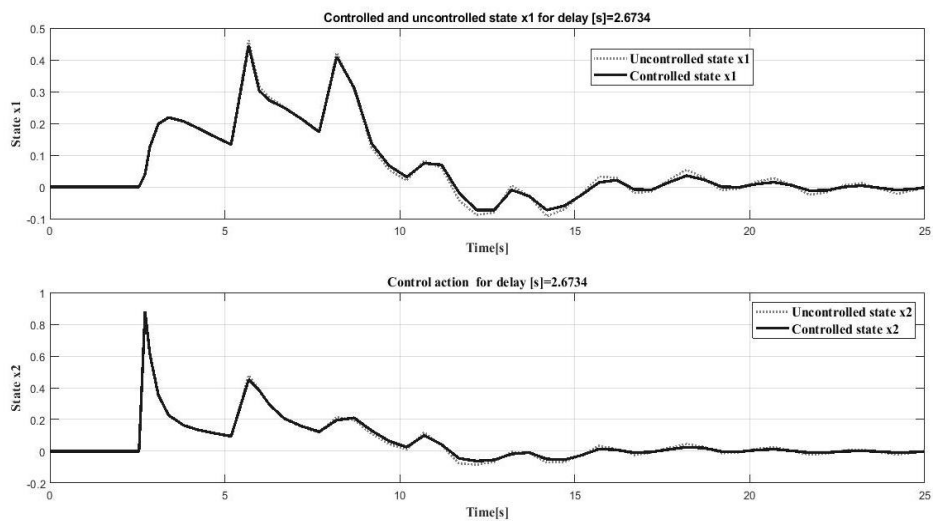

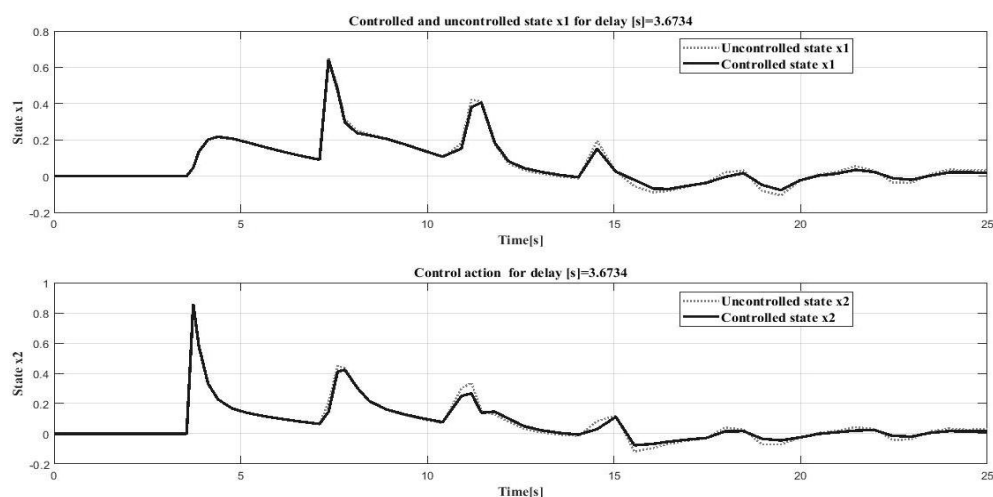**Fig. 5. Controlled and uncontrolled states with delay h $= 2.6527$**

Fig. 6. Controlled and uncontrolled states with delay $h = 3.6527$

The simulation outputs show different values of delays within the delay and outside the delay bounds. The effects of the time delay on the performance are analyzed for both the controlled and uncontrolled system. The simulations were caried out in SIMULINK with default parameter settings. Figs. 1, 2, 4 and 5 depicts the simulation of the system carried out within the delay bounds. That is, *h = 0.9920* and *h = 2.3718* respectively for Example 4.1, and *h = 1.8500* and *h = 2.6734* respectively for Example 4.3, while Figs. 3 and Fig. 6 shows when the delay is outside the range, that is *h = 3.3718* for Example 4.1 and *h = 3.6734* for Example 4.3. It was shown that settling time is faster as the delay increases within the bounds, see Figs. 2 and 5, more oscillations were observed in Figs. 1 and 4 that is, the states approach zero as the delay increases within the bound but increased oscillations were observed outside the delay bounds as shown in Figs. 3 and 6.

Competing Interests

Authors have declared that no competing interests exist.

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