

# Pyroelectric Plate with Magnetoelectric Effect

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## Abstract

In dynamic problems the electric and magnetic fields are inseparable. At the same time, a multitude of electrostatic and magnetostatic effects permit mutually independent description. This separation appears to be possible and thermodynamically consistent when the bulk energy density depends only on the polarization density or, alternatively, on the magnetization density. However, when the bulk energy density depends simultaneously on the both densities, then, the electrostatic and magnetostatic effects should be studied together. There appear interesting cross-effects; among those are the change of the internal electrostatic field inside a specimen under the influence of the external magnetic fields, and vice versa. Below, in the framework of thermodynamic approach the boundary value problem for magnetoelectric plate is formulated and analyzed. The exact solution is established for the isotropic pyroelectric plate.

## Keywords

Electricity and Magnetism, Pyroelectrics and Pyromagnetics, Thermodynamics of Electromagnetostatics, Exact Solutions, Validation and Verification

## 1. Introduction

As compared with nonpolarizable substances the bulk energy density of polarizable ones depends upon the additional macroscopic variable—the polarization density. Similarly, as compared with nonmagnetizable materials the bulk energy density of magnetizable ones depends upon the additional macroscopic variable—the magnetization density. More recently [1], in engineering and physics, there appeared interest in materials with the energy densities depending simultaneously on the polarization density and the magnetization density. Their main features and useful properties consist in the possibility of changing the magnetic

field and the magnetization density inside specimen by means of changing the external electrostatic field, and vice versa.

Thermodynamic (phenomenological) methods allow us to describe such phenomena by making minimum macroscopic assumptions and by introducing minimum additional material constants. Of course, some deep physical mechanisms, lying behind such possibilities, remain hidden in the phenomenological description, and they require further microscopic analysis. These microscopic aspects will not be discussed in this paper. Our analysis in this paper, focuses on the macroscopic (thermodynamic) analysis.

The readers, interested in microscopic aspects of magnetoelectric effects, are referred to the publications [2]-[16] and further references therein.

The closed master systems for analysis of magnetoelectric effects lead to non-linear boundary value problems. In general, the analysis of these problems requires specific computer coding. An essential part of the computer-based approach relies on the procedures of validation and verification. The validation and verification require establishing exact solutions of the corresponding boundary value problems. We establish some explicit solutions of the relevant boundary value problems which can be used for the validation and verification purposes.

## 2. The Basic Model

We accept the standard static bulk relationships of electrostatics and magnetostatics:

$$D^i \equiv E^i + 4\pi P^i \quad (1.1)$$

and

$$B^i \equiv H^i + 4\pi M^i, \quad (1.2)$$

In Equations (1.1), (1.2),  $E^i$  is the electrostatic field,  $P^i$  is the polarization density vector,  $D^i$  is the electrostatic displacement vector; similarly,  $H^i$  is the magnetic field,  $M^i$  the magnetization density vector,  $B^i$  the magnetic induction vector.

The electrostatics and magnetostatics bulk fields  $E^i$  and  $H^i$  have the potentials  $\varphi$  and  $\phi$ , respectively:

$$E_i = -\nabla_i \varphi \quad (1.3)$$

$$H_i = -\nabla_i \phi \quad (1.4)$$

Two other electrostatics and magnetostatics equations read

$$\nabla_i (E^i + 4\pi P^i) = 0 \quad (1.5)$$

and

$$\nabla_i (H^i + 4\pi M^i) = 0, \quad (1.6)$$

respectively.

The bulk equations of electrostatics and magnetostatics (1.1) - (1.6) should be

amended with the boundary conditions:

$$[\varphi]_-^+ = 0 \quad (1.7)$$

$$[-\nabla_i \varphi]_-^+ N^i + 4\pi [P_i]_-^+ N^i = 0 \quad (1.8)$$

$$[\phi]_-^+ = 0 \quad (1.9)$$

$$[-\nabla_i \phi]_-^+ N^i + 4\pi [M_i]_-^+ N^i = 0 \quad (1.10)$$

In the Equations (1.8), (1.10),  $N^i$  is a unit normal to the corresponding interface.

Let us turn now to the equations, relating to the thermodynamic of electricity and magnetism. Clearly, some electrostatic and magnetostatic variables should be added to the classical thermodynamic variables of liquid substances. To avoid possible misinterpretation, let us remind some facts from classical thermodynamics of compressible liquids. It is worth of reminding that any state of a two-parameter substance is fixed by the value of any two parameters; for instance, it can be a pair of specific volume  $v$  and entropy  $\sigma$ , or a pair of pressure  $p$  – and absolute temperature  $T$ , and so on. Such substances can be coined as thermodynamic substances with two degrees of freedom.

There is though one more question: which function or set of functions fully characterizes *all* states of the substance under study? This question should be clearly distinguished from the question about the number of degrees of freedom. The answer to the question about the full thermodynamic description of the substance is associated with the notion of the canonically associated thermodynamic potentials. Namely, each admissible pair of thermodynamic variables has the canonically associated with this pair thermodynamic potential; for instance, the internal energy potential  $e$  is canonically associated with the pair  $(v, \sigma)$ , the pair  $(v, T)$  is canonically associated with the free energy potential  $\psi(v, T)$ , and the pair  $(p, T)$  is canonically associated with the Grand thermodynamic potential  $G(p, T)$ , and so on. We remind that the choice of the set, including basic variables  $(v, \sigma)$  with the associated basic thermodynamic potential  $e(v, \sigma)$  contains the *full* information not only about particular state of the substance under discussion. This “fullness” means that any other thermodynamic variables and potential can be calculated with the help of algebraic operations and differentiation of the function  $e(v, \sigma)$ , *i.e.*, without using any additional physical measurements and hypotheses.

All the canonically associated sets are equivalent each other from the standpoint of the amount of full thermodynamic information that they contain. But this is mostly *mathematical*, logical equivalence. The relationships themselves, however, may be quite different for different equivalent sets. Depending on the particular systems and applications, different basic thermodynamic relationships can be considerably simplified by choosing the appropriate set of the basic thermodynamic variables and the potentials. For instance, when dealing with isothermal systems at fixed temperature, the free energy thermodynamic poten-

tial  $\psi(v, T)$  is much more convenient than the internal energy potential  $e(v, \eta)$ , which, in turn, is more convenient when considering adiabatic processes. Moreover, different mathematically equivalent sets may lead to different (and non-equivalent) further extensions and generalizations when considering essentially new applications.

When considering polarizable and magnetizable substances the number of thermodynamic degrees of freedom should be amended by inclusion also the polarization density  $P^i$ , or the magnetization density  $M^i$ , or both. These variables appear also in the corresponding internal energy function  $e(v, \sigma, P^i, M^i)$ , the free energy function  $\psi(v, T, P^i, M^i)$ , etc.

According to electrostatics, magnetostatics, and thermodynamics, the following thermodynamic relationships between the fields  $E_i$  and  $H_i$ , from the one hand, and the thermodynamic energy potential functions hold:

$$E_i(v, T, P^i, M^i) = \psi_{P^i}(v, T, P^i, M^i) \quad (1.11)$$

and

$$H_i(v, T, P^i, M^i) = \psi_{M^i}(v, T, P^i, M^i) \quad (1.12)$$

Also, we get the relationships

$$E_i(v, \sigma, P^i, M^i) = e_{P^i}(v, \sigma, P^i, M^i) \quad (1.13)$$

and

$$H_i(v, \sigma, P^i, M^i) = e_{M^i}(v, \sigma, P^i, M^i) \quad (1.14)$$

The relationship (1.11), (1.12) are considerably more convenient, than (1.13), (1.14) when dealing with the systems maintained at fixed temperature (the so-called systems in thermostat).

When considering systems in thermostat, the absolute temperature appears to be just a fixed parameter and it can be just omitted from the list of independent variables.

When the deformability or compressibility of the substance can be ignored, the variable  $v$  can be omitted from the list of thermodynamic variables. In the following, we limit ourselves with the systems in thermostat and ignore the parameters  $v$  and the absolute temperature. Thus, our main thermodynamic potentials will be  $\psi(P^i, M^i)$ , and the thermodynamic bulk identities (1.11), (1.12) read

$$E_i(P^i, M^i) = \psi_{P^i}(P^i, M^i) \quad (1.15)$$

and

$$H_i(P^i, M^i) = \psi_{M^i}(P^i, M^i) \quad (1.16)$$

### 3. The Boundary Value Problems for Magnetoelectric Solids

Using the relationships (1.15), (1.16), we arrive at the following relationships of the electric  $D^i$  and the magnetic  $B^i$  inductions:

$$D^i \equiv \psi_{P^j} z^{ij} + 4\pi P^i \tag{2.1}$$

and

$$B^i \equiv \psi_{M^j} z^{ij} + 4\pi M^i, \tag{2.2}$$

respectively.

Using relationships (2.1), (2.2), we can rewrite the bulk Equations (1.5), (1.6) as

$$\nabla_i (\psi_{P^j} z^{ij} + 4\pi P^i) = 0 \tag{2.3}$$

and

$$\nabla_i (\psi_{M^j} z^{ij} + 4\pi M^i) = 0, \tag{2.4}$$

respectively.

Equations (2.3), (2.4) can be rewritten as follows

$$\psi_{P^j P^k} \nabla^j P^k + \psi_{P^j M^k} \nabla^j M^k + 4\pi \nabla_j P^j = 0 \tag{2.5}$$

and

$$\psi_{M^j M^k} \nabla^j M^k + \psi_{M^j P^k} \nabla^j P^k + 4\pi \nabla_j M^j = 0 \tag{2.6}$$

The bulk Equations (1.15), (1.16) imply

$$\nabla_i \phi = \psi_{P^i} \tag{2.7}$$

and

$$\nabla_i \phi = \psi_{M^i} \tag{2.8}$$

### 4. The Combined Action of Electrostatic and Magnetostatic Fields

Resolving the pair (1.15), (1.16) with respect to  $P^i$  and  $M^i$ , we get

$$P^i \equiv \Pi^i (E^m, H^n), \quad M^i \equiv \Theta^i (E^m, H^n) \tag{3.1}$$

Inserting (3.1) in the electrostatic and magnetostatic equations, we get

$$\nabla_i (-E_j z^{ij} + 4\pi \Pi^i (E^m, H^n)) = 0 \tag{3.2}$$

and

$$\nabla_i (-H_j z^{ij} + 4\pi \Theta^i (E^m, H^n)) = 0 \tag{3.3}$$

Now, using (1.7), (1.8), we can rewrite (4.4), (4.5) as follows:

$$\nabla_i (\nabla^i \phi + 4\pi \Pi^i (-\nabla^m \phi, -\nabla^n \phi)) = 0 \tag{3.4}$$

and

$$\nabla_i (\nabla^i \phi + 4\pi \Theta^i (-\nabla^m \phi, -\nabla^n \phi)) = 0 \tag{3.5}$$

The bulk Equations (3.4), (3.5) should be amended with the following boundary conditions at the surfaces of discontinuity:

$$[\varphi]_-^+ = 0 \quad (3.6)$$

$$\left[ -\nabla_i \varphi + 4\pi \Pi_i (-\nabla^m \varphi, -\nabla^m \phi) \right]_-^+ N^i = 0 \quad (3.7)$$

$$[\phi]_-^+ = 0 \quad (3.8)$$

$$\left[ -\nabla_i \phi + 4\pi \Theta_i (-\nabla^m \varphi, -\nabla^n \phi) \right]_-^+ N^i = 0 \quad (3.9)$$

The system (3.4) - (3.9) comprise closed system of equations and boundary conditions free of any assumptions of smallness.

## 5. Linear Pyroelectric and Pyromagnetic Substances

The system (3.4) - (3.9) is nonlinear and it is rarely can be handled analytically. The system (3.4) - (3.9) is more convenient when we consider the model such that

$$\psi(P^i, M^i) = \mu T_i P^i + \eta \Upsilon_i M^i + \alpha \frac{1}{2} z_{ij} P^i P^j + \beta \frac{1}{2} z_{ij} M^i M^j + \gamma z_{ij} P^i M^j, \quad (4.1)$$

where  $\alpha, \beta, \gamma, \mu, \eta$  are certain constants, and  $T_i, \Upsilon_i$  are certain vectors. The vectors  $\mu T_i P^i$  and  $\eta \Upsilon_i M^i$  describe the pyroelectric and pyromagnetic effects, respectively. The term  $\gamma z_{ij} P^i M^j$  describes the static interaction between the polarization and magnetization.

The potential (4.1) implies the following relationships:

$$\begin{aligned} \psi_{P^i P^j} &= \alpha z_{ij} = \text{const}, \quad \psi_{M^i M^j} = \beta z_{ij} = \text{const}, \\ \psi_{P^i M^j} &= \psi_{M^i P^j} = \gamma z_{ij} = \text{const} \end{aligned} \quad (4.2)$$

Further differentiation of (4.1), being combined with thermodynamic identities (1.15), (1.16), implies the following expressions for the electric and magnetic fields:

$$E_i = \mu T_i + \alpha P_i + \gamma M_i \quad (4.3)$$

and

$$H_i = \eta \Upsilon_i + \beta M_i + \gamma P_i \quad (4.4)$$

Resolving the pair (4.3), (4.4) with respect to polarization  $P_i$  and magnetization  $M_i$ , we get the relationships:

$$P_i = \frac{\beta E_i - \gamma H_i + \gamma \eta \Upsilon_i - \beta \mu T_i}{\alpha \beta - \gamma^2} \quad (4.5)$$

and

$$M_i = \frac{\alpha H_i - \gamma E_i - \alpha \eta \Upsilon_i + \gamma \mu T_i}{\alpha \beta - \gamma^2} \quad (4.6)$$

The relationships (4.5), (4.6) are implied by the following chains:

$$\begin{aligned} E_i &= \alpha P_i + \gamma M_i + \mu T_i \rightarrow E_i - \mu T_i - \alpha P_i = \gamma M_i \\ \rightarrow H_i &= \eta \Upsilon_i + \beta M_i + \gamma P_i \rightarrow H_i - \eta \Upsilon_i - \gamma P_i = \beta M_i \\ \rightarrow \gamma H_i - \gamma \eta \Upsilon_i - \gamma^2 P_i &= \gamma \beta M_i \\ \rightarrow \gamma H_i - \gamma \eta \Upsilon_i - \gamma^2 P_i &= \beta (E_i - \mu T_i - \alpha P_i) \end{aligned}$$

$$\begin{aligned} &\rightarrow \gamma H_i - \gamma \eta \Upsilon_i - \gamma^2 P_i = \beta E_i - \beta \mu T_i - \alpha \beta P_i \\ &\rightarrow (\alpha \beta - \gamma^2) P_i = \beta E_i - \gamma H_i + \gamma \eta \Upsilon_i - \beta \mu T_i \\ &\rightarrow P_i = \frac{\beta E_i - \gamma H_i + \gamma \eta \Upsilon_i - \beta \mu T_i}{\alpha \beta - \gamma^2} \end{aligned}$$

and

$$\begin{aligned} H_i &= \eta \Upsilon_i + \beta M_i + \gamma (\alpha^{-1} E_i - \alpha^{-1} \mu T_i - \alpha^{-1} \gamma M_i) \\ &\rightarrow H_i - \gamma \alpha^{-1} E_i = \eta \Upsilon_i - \gamma \alpha^{-1} \mu T_i + (\beta - \alpha^{-1} \gamma^2) M_i \\ &\rightarrow \alpha H_i - \gamma E_i = \alpha \eta \Upsilon_i - \gamma \mu T_i + (\alpha \beta - \gamma^2) M_i \\ &\rightarrow \frac{\alpha H_i - \gamma E_i - \alpha \eta \Upsilon_i + \gamma \mu T_i}{\alpha \beta - \gamma^2} = M_i \end{aligned}$$

respectively.

Relationships (4.5), (4.6) lead to the following relationships of the electric displacement  $D^i$  and magnetic induction  $B^i$ :

$$D_i = \frac{E_i (\alpha \beta - \gamma^2 + 4\pi \beta) - 4\pi \gamma H_i}{\alpha \beta - \gamma^2} + 4\pi \frac{\gamma \eta \Upsilon_i - \beta \mu T_i}{\alpha \beta - \gamma^2} \tag{4.7}$$

and

$$B_i = \frac{H_i (\alpha \beta - \gamma^2 + 4\pi \alpha) - 4\pi \gamma E_i}{\alpha \beta - \gamma^2} + 4\pi \frac{\gamma \mu T_i - \alpha \eta \Upsilon_i}{\alpha \beta - \gamma^2}, \tag{4.8}$$

respectively.

The relationships (4.7), (4.8) are implied by the following chains:

$$\begin{aligned} D_i &= E_i + 4\pi P_i = E_i + 4\pi \frac{\beta E_i - \gamma H_i + \gamma \eta \Upsilon_i - \beta \mu T_i}{\alpha \beta - \gamma^2} \\ &= E_i + 4\pi \frac{\beta E_i - \gamma H_i}{\alpha \beta - \gamma^2} + 4\pi \frac{\gamma \eta \Upsilon_i - \beta \mu T_i}{\alpha \beta - \gamma^2} \\ &= \frac{E_i (\alpha \beta - \gamma^2 + 4\pi \beta) - 4\pi \gamma H_i}{\alpha \beta - \gamma^2} + 4\pi \frac{\gamma \eta \Upsilon_i - \beta \mu T_i}{\alpha \beta - \gamma^2} \\ &\rightarrow D_i = \frac{E_i (\alpha \beta - \gamma^2 + 4\pi \beta) - 4\pi \gamma H_i}{\alpha \beta - \gamma^2} + 4\pi \frac{\gamma \eta \Upsilon_i - \beta \mu T_i}{\alpha \beta - \gamma^2} \end{aligned}$$

and

$$\begin{aligned} B_i &= H_i + 4\pi M_i = H_i + 4\pi \frac{\alpha H_i - \gamma E_i - \alpha \eta \Upsilon_i + \gamma \mu T_i}{\alpha \beta - \gamma^2} \\ &= H_i + 4\pi \frac{\alpha H_i - \gamma E_i}{\alpha \beta - \gamma^2} + 4\pi \frac{\gamma \mu T_i - \alpha \eta \Upsilon_i}{\alpha \beta - \gamma^2} \\ &= \frac{H_i (\alpha \beta - \gamma^2 + 4\pi \alpha) - 4\pi \gamma E_i}{\alpha \beta - \gamma^2} + 4\pi \frac{\gamma \mu T_i - \alpha \eta \Upsilon_i}{\alpha \beta - \gamma^2} \\ B_i &= \frac{H_i (\alpha \beta - \gamma^2 + 4\pi \alpha) - 4\pi \gamma E_i}{\alpha \beta - \gamma^2} + 4\pi \frac{\gamma \mu T_i - \alpha \eta \Upsilon_i}{\alpha \beta - \gamma^2} \end{aligned}$$

Using (4.7), (4.8), we arrive at the following bulk equations for the electrostat-

ic and magnetostatic potentials:

$$\nabla_i D^i = 0 \rightarrow \frac{\alpha\beta - \gamma^2 + 4\pi\beta}{\alpha\beta - \gamma^2} \nabla^i \nabla_i \varphi - \frac{4\pi\gamma}{\alpha\beta - \gamma^2} \nabla^i \nabla_i \phi = 0 \tag{4.9}$$

and

$$\nabla^i B_i = 0 \rightarrow \frac{\alpha\beta - \gamma^2 + 4\pi\alpha}{\alpha\beta - \gamma^2} \nabla_i \nabla^i \phi - \frac{4\pi\gamma}{\alpha\beta - \gamma^2} \nabla_i \nabla^i \varphi = 0 \tag{4.10}$$

The associated boundary conditions read

$$\left[ -\frac{\alpha\beta - \gamma^2 + 4\pi\beta}{\alpha\beta - \gamma^2} \nabla_i \varphi + \frac{4\pi\gamma}{\alpha\beta - \gamma^2} \nabla_i \phi + 4\pi \frac{\gamma\eta\Upsilon_i - \beta\mu T_i}{\alpha\beta - \gamma^2} \right]_+^- N^i = 0 \tag{4.11}$$

and

$$\left[ -\frac{\alpha\beta - \gamma^2 + 4\pi\alpha}{\alpha\beta - \gamma^2} \nabla_i \phi + \frac{4\pi\gamma}{\alpha\beta - \gamma^2} \nabla_i \varphi + 4\pi \frac{\gamma\mu T_i - \alpha\eta\Upsilon_i}{\alpha\beta - \gamma^2} \right]_+^- N^i = 0 \tag{4.12}$$

Considering the uniform isotropic layer made of general linear pyroelectric substance we arrive at the following relationships for the electrostatic and magnetostatic fields inside the layer:

$$E^i = 4\pi \frac{4\pi\gamma(\gamma\mu T_N - \alpha\eta\Upsilon_N) - (\beta\mu T_N - \gamma\eta\Upsilon_N)(\alpha\beta - \gamma^2 + 4\pi\alpha)}{16\pi^2\gamma^2 - (\alpha\beta - \gamma^2 + 4\pi\alpha)(\alpha\beta - \gamma^2 + 4\pi\beta)} N^i \tag{4.13}$$

and

$$H^i = 4\pi \frac{4\pi\gamma(\gamma\eta\Upsilon_N - \beta\mu T_N) + (\gamma\mu T_N - \alpha\eta\Upsilon_N)(\alpha\beta - \gamma^2 + 4\pi\beta)}{16\pi^2\gamma^2 - (\alpha\beta - \gamma^2 + 4\pi\alpha)(\alpha\beta - \gamma^2 + 4\pi\beta)} N^i \tag{4.14}$$

where  $T_N$  and  $\Upsilon_N$  are defined as

$$T_N \equiv T^i N_i, \quad \Upsilon_N \equiv \Upsilon^i N_i \tag{4.15}$$

### 6. The Vanishing Connection Constant

At the vanishing connection constant  $\gamma$ , the Equations (4.9), (4.10) imply

$$\nabla^i \nabla_i \varphi = 0 \tag{5.1}$$

and

$$\nabla_i \nabla^i \phi = 0, \tag{5.2}$$

respectively.

The potentials continuity conditions

$$[\varphi]_+^- = 0 \tag{5.3}$$

and

$$[\phi]_+^- = 0, \tag{5.4}$$

should be amended with the following implications of the boundary conditions (4.11), (4.12):



$$\left[ \frac{\alpha + 4\pi}{\alpha} \nabla_i \varphi + 4\pi \frac{\mu T_i}{\alpha} \right]_+ N^i = 0 \quad (5.5)$$

and

$$\left[ \frac{\beta + 4\pi}{\beta} \nabla_i \phi + 4\pi \frac{\eta Y_i}{\beta} \right]_+ N^i = 0, \quad (5.6)$$

respectively.

The solution (4.13), (4.14) takes on the following form:

$$E^i = \frac{4\pi\mu}{\alpha + 4\pi} T_k N^k N^i \quad (5.7)$$

and

$$H^i = \frac{4\pi\eta}{\beta + 4\pi} Y_k N^k N^i \quad (5.8)$$

## 7. Discussion and Conclusion

One of possible thermodynamically consistent models was suggested and analyzed theoretically. The model is based on the traditional ideas of electrostatics and magnetostatics. In addition, our model is based on the free energy density function  $\psi(v, T, P^i, M^i)$  or the internal energy density  $e(v, \sigma, P^i, M^i)$ . The polarization  $P^i$  and magnetization densities  $M^i$ , the specific volume  $v$  together with the absolute temperature  $T$  can be chosen as independent thermodynamic variables when considering systems in thermostat. Thermodynamic arguments lead to the thermodynamic identities (1.11), (1.12) for the electric and magnetic fields. Equivalently, we can choose the independent variables  $(v, \sigma, P^i, M^i)$  and the internal energy density  $e(v, \sigma, P^i, M^i)$  instead of the set  $(v, T, P^i, M^i)$  and the free energy density  $\psi(v, T, P^i, M^i)$ . In this case we arrive at the thermodynamic identities (1.13), (1.14) instead of (1.11), (1.12).

We reduce the general boundary value problem of electrostatics/magnetostatics to the system with 2 unknown electrostatic  $\varphi$  and magnetostatic  $\phi$  potentials, respectively. The system (3.4), (3.5) appears to be deeply nonlinear and hardly tractable analytically.

We proceed by applying the general identities to the special case of the quadratic free energy density function. That approximation permits to reduce the nonlinear system (3.4), (3.5) to the linear system (4.9), (4.10) which is much more tractable analytically. The system (4.9), (4.10) should be amended with the boundary conditions (4.11), (4.12). The system (4.9) - (4.12) is applied to the analysis of a uniform isotropic layer, made of general linear pyroelectric substances. The exact solution leads to the relationships (4.13), (4.14) for the electrostatic and magnetostatic fields inside the layer. This exact solution can be recommended for the validation/verification purposes when developing numerical codes.

## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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