Hindawi
Abstract and Applied Analysis
Volume 2024, Article ID 6682607, 9 pages
https://doi.org/10.1155/2024/6682607

Research Article

Oscillation of Fourth-Order Nonlinear Semi-Canonical Neutral Difference Equations via Canonical Transformations

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Received 28 July 2023; Revised 17 November 2023; Accepted 9 December 2023; Published 29 January 2024

Academic Editor: Agacik Zafer

The authors present a new technique for transforming fourth-order semi-canonical nonlinear neutral difference equations into canonical form. This greatly simplifies the examination of the oscillation of solutions. Some new oscillation criteria are established by comparison with first-order delay difference equations. Examples are provided to illustrate the significance and novelty of the main results. The results are new even for the case of nonneutral difference equations.

1. Introduction

In recent years, there has been an increasing interest in studying the oscillatory properties of difference equations; see for example, the monographs [1–3]. This interest is motivated by the importance of difference equations in modeling real-world problems and in the numerical solution of differential equations. In particular, the oscillatory and asymptotic behavior of solutions of fourth-order delay and neutral type difference equations have received great attention in the last few years; see, for example, the papers [3–23].

In view of the above facts, in this paper, we consider the fourth-order nonlinear neutral delay difference equation:

$$D_4 z(n) + q(n) x^{\alpha}(n-\tau) = 0, n \geq n_0 > 0, \tag{1}$$

where $D_0 z(n) = z(n)$, $D_i z(n) = a_i(n) \Delta(D_{i-1} z(n))$, $i = 1, 2, 3$, $D_4 z(n) = \Delta(D_3 z(n))$, and $z(n) = x(n) + p(n) x(n-\sigma)$. Throughout the paper, we assume that:

(H_1) $\{a_j(n)\}$, $j = 1, 2, 3$, are positive sequences of real numbers;
(H_2) $\{p(n)\}$ and $\{q(n)\}$ are nonnegative real sequences with $0 \leq p(n) < 1$;
(H_3) α is a ratio of odd positive integers and σ and τ are positive integers.

By a solution of (1), we mean a real sequence $\{x(n)\}$ satisfying Equation (1) for all $n \geq n_0$. We consider only such solutions that are nontrivial for all large n. A solution of (1) is called *nonoscillatory* if it is eventually positive or eventually negative; otherwise it is called *oscillatory*. If all solutions are oscillatory, then the equation itself called oscillatory.

In reviewing the literature, it is seen that the most known results are for Equation (1) when it is in *canonical form*, that is,

$$\sum_{s=n_0}^{\infty} \frac{1}{a_j(s)} = \infty \text{ for } j = 1, 2, 3. \tag{2}$$

In order to clarify and facilitate our discussion, we introduce the following terminology and classification of Equation (1). We define:

$$A_i(n) = \sum_{s=n}^{\infty} \frac{1}{a_i(s)}, \text{ for } i = 1, 2, 3, \tag{3}$$

and we will say that Equation (1) is in *canonical form* if:

$$A_i(n_0) = \infty, \ i = 1, 2, 3, \tag{4}$$

and it is in *noncanonical form* if:

$$A_i(n_0) < \infty, \ i = 1, 2, 3. \tag{5}$$

In addition, we will say that Equation (1) is in *semi-canonical form* if either:

$$A_1(n_0) = A_2(n_0) = \infty \text{ and } A_3(n_0) < \infty, \tag{6}$$

or

$$A_1(n_0) = \infty, A_2(n_0) < \infty, \text{ and } A_3(n_0) = \infty, \tag{7}$$

or

$$A_1(n_0) < \infty \text{ and } A_2(n_0) = A_3(n_0) = \infty. \tag{8}$$

Finally, we will say that Equation (1) is in *semi-noncanonical form* if either:

$$A_1(n_0) = \infty, A_2(n_0) < \infty, \text{ and } A_3(n_0) < \infty, \tag{9}$$

or

$$A_1(n_0) < \infty, A_2(n_0) = \infty, \text{ and } A_3(n_0) < \infty, \tag{10}$$

or

$$A_1(n_0) < \infty, A_2(n_0) < \infty, \text{ and } A_3(n_0) = \infty. \tag{11}$$

Note. When we refer to Equation (1) being in canonical, noncanonical, semi-canonical, or semi-noncanonical form, we will use this same terminology to describe the operator $D_4 z(n)$ in the equation as well.

In the papers [4, 8, 10, 15, 16, 21–23], the authors studied Equation (1) in case (6) or (7) holds, that is Equation (1) is in *semi-canonical form*. In these cases, the positive solutions of the equation satisfy one of four possible cases, and each of these have to be eliminated to obtain the oscillation of all solutions. Therefore, our aim in this paper is to first transform Equation (1) in cases (6) or (7) into canonical form. This greatly simplifies the examination of the equation since, in this case, the positive solutions are one of only two types. We then apply techniques known for canonical equations to obtain oscillation criteria for Equation (1). Examples illustrating the significance and novelty of our main results are provided. Note that, our results established here are new for nonneutral difference equations as well.

2. Main Results

In this section, we study the oscillatory behavior of Equation (1) in case (6) or (7) holds.

2.1. The Case Where (6) Holds. We begin by defining:

$$b_1(n) = a_1(n), \ b_2(n) = \frac{a_2(n)}{A_3(n)}, \tag{12}$$

$$b_3(n) = a_3(n)A_3(n)A_3(n+1), \ Q(n) = A_3(n+1)q(n). \tag{13}$$

Theorem 1. *Assume that*

$$\sum_{n=n_0}^{\infty} \frac{A_3(n)}{a_2(n)} = \infty. \tag{14}$$

Then, the semi-canonical operator $D_4 z(n)$ can be written in canonical form as follows:

$$D_4 z(n) = \frac{1}{A_3(n+1)} \Delta(b_3(n)\Delta(b_2(n)\Delta(b_1(n)\Delta z(n)))). \tag{15}$$

Proof. A direct calculation shows that:

$$\begin{aligned}
&\frac{1}{A_3(n+1)} \Delta(b_3(n)\Delta(b_2(n)\Delta(b_1(n)\Delta z(n)))) \\
&= \frac{1}{A_3(n+1)} \Delta\left(b_3(n)\Delta\left(\frac{a_2(n)}{A_3(n)}\Delta(b_1(n)\Delta z(n))\right)\right) \\
&= \frac{1}{A_3(n+1)} \Delta\left(\frac{b_3(n)a_3(n)A_3(n)\Delta(a_2(n)\Delta(b_1(n)\Delta z(n))) + b_3(n)a_2(n)\Delta(b_1(n)\Delta z(n))}{A_3(n)A_3(n+1)a_3(n)}\right) \\
&= \frac{1}{A_3(n+1)} \Delta(A_3(n)a_3(n)\Delta(a_2(n)\Delta(b_1(n)\Delta z(n))) + a_2(n)\Delta(b_1(n)\Delta z(n))) \\
&= \Delta(a_3(n)\Delta(a_2(n)\Delta(b_1(n)\Delta z(n)))) - \Delta(a_2(n)\Delta(b_1(n)\Delta z(n)))/A_3(n+1) \\
&\quad + \Delta(a_2(n)\Delta(b_1(n)\Delta z(n)))/A_3(n+1) \\
&= D_4 z(n).
\end{aligned} \tag{16}$$

Furthermore,

$$\sum_{n=n_0}^{\infty}\frac{1}{b_3(n)}=\sum_{n=n_0}^{\infty}\frac{1}{a_3(n)A_3(n)A_3(n+1)}=\sum_{n=n_0}^{\infty}\Delta\left(\frac{1}{A_3(n)}\right)$$
$$=\lim_{n\to\infty}\frac{1}{A_3(n)}-\frac{1}{A_3(n_0)}=\infty, \tag{17}$$

and

$$\sum_{n=n_0}^{\infty}\frac{1}{b_2(n)}=\sum_{n=n_0}^{\infty}\frac{A_3(n)}{a_2(n)}=\infty, \tag{18}$$

by (14). Also,

$$\sum_{n=n_0}^{\infty}\frac{1}{b_1(n)}=\sum_{n=n_0}^{\infty}\frac{1}{a_1(n)}=\infty, \tag{19}$$

by (6). Hence, the right hand side of (15) is in canonical form. This completes the proof of the theorem. □

From the above theorem, we obtain the following corollary.

Corollary 1. *Let (14) hold. Then $x(t)$ is a solution of Equation (1) if and only if it is also a solution of the canonical equation:*

$$\overline{D}_4 z(n)+Q(n)x^{\alpha}(n-\tau)=0, \tag{20}$$

where $\overline{D}_0 z(n)=z(n)$, $\overline{D}_i z(n)=b_i(n)\Delta(\overline{D}_{i-1}z(n))$, $i=1,2,3$, and $\overline{D}_4 z(n)=\Delta(\overline{D}_3 z(n))$.

The next lemma gives us the classification of positive nonoscillatory solutions of Equation (20).

Lemma 1. *Let $\{x(n)\}$ be an eventually positive solution of (20). Then the corresponding sequence $\{z(n)\}$ is also eventually positive, and exactly one of the following statements holds:*

(I) $\overline{D}_1 z(n)>0,\ \overline{D}_2 z(n)<0,\ \overline{D}_3 z(n)>0,\ \overline{D}_4 z(n)\le 0$;
(II) $\overline{D}_1 z(n)>0,\ \overline{D}_2 z(n)>0,\ \overline{D}_3 z(n)>0,\ \overline{D}_4 z(n)\le 0$,

for sufficiently large n.

Proof. The proof is similar to that of [11, Lemma 3] and so we omit the details. □

Lemma 2. *Let $\{x(n)\}$ be an eventually positive solution of (E_1). Then:*

$$x(n)\ge(1-p(n))z(n-\sigma), \tag{21}$$

for all $n\ge n_1\ge n_0$.

Proof. From the definition of $z(n)$, we have:

$$x(n)=z(n)-p(n)x(n-\sigma)\ge z(n)-p(n)z(n-\sigma). \tag{22}$$

In view of Lemma 1, we see that $z(n)$ satisfies:

$$\overline{D}_1 z(n)=b_1(n)\Delta z(n)>0, \tag{23}$$

for both cases (I) and (II). This implies that $z(n)$ is increasing and using this in (22) yields:

$$x(n)\ \ge\ (1-p(n))z(n-\sigma). \tag{24}$$

This proves the lemma. □

We next obtain another oscillation result.

Theorem 2. *Let (14) hold. If*

$$\sum_{n=n_1}^{\infty}Q(n)(1-p(n-\tau))^{\alpha}=\infty, \tag{25}$$

then Equation (1) is oscillatory.

Proof. Let $\{x(n)\}$ be an eventually positive solution of (E). Then by Corollary 1, we see that $\{x(n)\}$ is also an eventually positive solution of (E_1). Then by Lemma 1, the sequence $\{z(n)\}$ satisfies either Case (I) or Case (II) for all $n\ge n_1\ge n_0$.

Now using (21) in (E_1), we obtain:

$$\overline{D}_4 z(n)+Q(n)(1-p(n-\tau))^{\alpha}z^{\alpha}(n-\tau-\sigma)\le 0,\ \ n\ge n_1. \tag{26}$$

In both cases $z(n)$ is increasing, so there exists a constant $M>0$ and an integer $n_2\ge n_1$ such that $z(n-\tau-\sigma)\ge M$ for $n\ge n_2$. Using this in Equation (26), we have

$$-\overline{D}_4 z(n)\ge M^{\alpha}Q(n)(1-p(n-\tau))^{\alpha}. \tag{27}$$

Summing Equation (27), from n_2 to n gives:

$$M^{\alpha}\sum_{s=n_2}^{n}Q(s)(1-p(s-\tau))^{\alpha}\le\overline{D}_3 z(n_2)-\overline{D}_3 z(n+1)\le\overline{D}_3 z(n_2), \tag{28}$$

since in both cases $\overline{D}_3 z(n)>0$. This contradiction completes the proof of the theorem. □

Remark 1. Theorem 2 is independent of the values of α and the delay argument τ. Hence, it is applicable to linear, sublinear, or superlinear equations as well as to ordinary, delay, or advanced type difference equations.

Before we present our next results, we introduce the notation:

$$Q_1(n)=\left(\frac{1}{b_2(n)}\sum_{s=n}^{\infty}\frac{1}{b_3(s)}\sum_{u=s}^{\infty}Q(u)(1-p(u-\tau))^{\alpha}\right)\left(\sum_{s=n_*}^{n-\tau-\sigma-1}\frac{1}{b_1(s)}\right)^{\alpha}, \tag{29}$$

and

$$Q_2(n)= Q(n)(1-p(n-\tau))^{\alpha}\left(\sum_{s=n_*}^{n-\sigma-\tau-1}\frac{1}{b_1(s)}\sum_{u=s_*}^{s-1}\frac{1}{b_2(u)}\sum_{s_1=n_*}^{u-1}\frac{1}{b_3(s_1)}\right)^{\alpha}, \tag{30}$$

where $n_*\geq n_0$ is a sufficiently large integer.

Theorem 3. *Let (14) hold and assume that both of the first-order delay difference equations:*

$$\Delta w(n)+Q_1(n)w^{\alpha}(n-\tau-\sigma)=0, \tag{31}$$

and

$$\Delta w(n)+Q_2(n)w^{\alpha}(n-\tau-\sigma)=0, \tag{32}$$

are oscillatory. Then Equation (1) is oscillatory.

Proof. Let $x(n)$ be an eventually positive solution of (E), say $n\geq n_1\geq n_0$. Then by Corollary 1, we see that $\{x(n)\}$ is a positive solution of Equation (20), and by Lemma 1, the function $z(n)$ is positive and belongs to either Class (I) or Class (II). Moreover, using Equation (21) in (20), we obtain:

$$\overline{D}_4z(n)+Q(n)(1-p(n-\tau))^{\alpha}z^{\alpha}(n-\tau-\sigma)\leq 0, \tag{33}$$

for all $n\geq n_2\geq n_1+\tau+\sigma$.

First assume that $z(n)$ is in Class (I). Notice that $\overline{D}_1z(n)$ is decreasing, and we have:

$$z(n)\geq\sum_{s=n_2}^{n-1}\frac{1}{b_1(s)}b_1(s)\Delta z(s)\geq\overline{D}_1z(n)\sum_{s=n_2}^{n-1}\frac{1}{b_1(s)}. \tag{34}$$

Summing Equation (33), from n to ∞, we obtain:

$$\overline{D}_3z(n)\geq\sum_{s=n}^{\infty}Q(s)(1-p(s-\tau))^{\alpha}z^{\alpha}(s-\tau-\sigma). \tag{35}$$

Since $z(n-\tau-\sigma)$ is increasing, this implies:

$$\Delta\left(\overline{D}_2z(n)\right)\geq\frac{z^{\alpha}(n-\tau-\sigma)}{b_3(n)}\sum_{s=n}^{\infty}Q(s)(1-p(s-\tau))^{\alpha}. \tag{36}$$

Summing Equation (36), we find that:

$$-\Delta\left(\overline{D}_1z(n)\right)\geq\frac{z^{\alpha}(n-\tau-\sigma)}{b_2(n)}\sum_{s=n}^{\infty}\frac{1}{b_3(s)}\sum_{u=s}^{\infty}Q(u)(1-p(u-\tau))^{\alpha}. \tag{37}$$

Using Equation (34) in (37) gives:

$$-\Delta\left(\overline{D}_1z(n)\right)\geq\left(\overline{D}_1z(n-\tau-\sigma)\right)^{\alpha}Q_1(n). \tag{38}$$

Hence, the sequence $w(n)=\overline{D}_1z(n)$ is a positive solution of the delay difference inequality:

$$\Delta w(n)+Q_1(n)w^{\alpha}(n-\tau-\sigma)\leq 0. \tag{39}$$

By Grace and Graef [7, Lemma 3], we see that the associated delay difference Equation (31) also has a positive solution, which is a contradiction.

Next, assume that $z(n)$ belongs to Class (II). Since $\overline{D}_3z(n)$ is decreasing, we have:

$$\overline{D}_2z(n)\geq\sum_{s=n_2}^{n-1}\frac{1}{b_3(s)}\overline{D}_3z(s)\geq\overline{D}_3z(n)\sum_{s=n_2}^{n-1}\frac{1}{b_3(s)}. \tag{40}$$

Summing the above inequality, we obtain:

$$\Delta z(n)\geq\overline{D}_3z(n)\frac{1}{b_1(n)}\sum_{s=n_2}^{n-1}\frac{1}{b_2(s)}\sum_{u=n_2}^{s-1}\frac{1}{b_3(u)}, \tag{41}$$

and summing again, we see that $w(n)=\overline{D}_3z(n)$ satisfies:

$$z(n)\geq w(n)\sum_{s=n_2}^{n-1}\frac{1}{b_1(s)}\sum_{u=n_2}^{s-1}\frac{1}{b_2(u)}\sum_{s_1=n_2}^{u-1}\frac{1}{b_3(s_1)}. \tag{42}$$

Using the last estimate in Equation (33) shows that $w(n)$ is a positive solution of the difference inequality:

$$\Delta w(n)+Q_2(n)w^{\alpha}(n-\tau-\sigma)\leq 0, \tag{43}$$

which implies that the corresponding difference Equation (32) also has a positive solution. In view of Grace and Graef [7, Lemma 3], this is again a contradiction and proves the theorem. □

Corollary 2. *Assume that, Equation (14) holds. If*

$$\liminf_{n\to\infty}\sum_{s=n-\tau-\sigma}^{n-1}H(s)\geq\left(\frac{\tau+\sigma}{\tau+\sigma+1}\right)^{\tau+\sigma+1},\ \text{for }\alpha=1, \tag{44}$$

or

$$\sum_{n=n_*}^{\infty} H(n) = \infty, \text{ for } 0<\alpha<1, \tag{45}$$

where

$$H(n) = \min\{Q_1(n), Q_2(n)\}, \tag{46}$$

then Equation (1) is oscillatory.

Proof. It is clear (e.g., [3, Theorem 7.6.1] and [13, Theorem 1], respectively) that conditions (44) and (45) ensure the oscillation of (31) and (32) in the cases $\alpha=1$ and $0<\alpha<1$, respectively. This proves the corollary. □

Corollary 3. *Assume that, Equation (14) holds. If $\alpha>1$ and there exists a constant $\lambda>1/(\tau+\sigma)\ln\alpha$ such that:*

$$\liminf_{n\to\infty} [H(s)\exp(-e^{\lambda n})]>0, \tag{47}$$

where $H(n)$ is defined as in Corollary 2, then Equation (1) is oscillatory.

The conclusion of this corollary follows from Theorem 3 and study by Tang [13, Theorem 2].

2.2. The Case Where (7) Holds. We set

$$d_1(n) = \frac{a_1(n)}{A_2(n)},\ d_2(n) = a_2(n)A_2(n)A_2(n+1), \text{ and}$$
$$d_3(n) = \frac{a_3(n)}{A_2(n+1)}, \tag{48}$$

for all $n\geq n_*\geq n_0$.

The following result is analogous to Theorem 1.

Theorem 4. *Assume that*

$$\sum_{n=n_0}^{\infty} \frac{A_2(n+1)}{a_3(n)} = \sum_{n=n_0}^{\infty} \frac{A_2(n)}{a_1(n)} = \infty. \tag{49}$$

Then the semi-canonical operator $D_4z(n)$ has the canonical representation:

$$D_4z(n) = \Delta\left(\frac{a_3(n)}{A_2(n+1)}\Delta\left(a_2(n)A_2(n)A_2(n+1)\Delta\left(\frac{a_1(n)}{A_2(n)}\Delta z(n)\right)\right)\right). \tag{50}$$

Proof. Taking the difference:

$$\begin{aligned}
&\Delta(d_3(n)\Delta(d_2(n)\Delta(d_1(n)\Delta z(n))))\\
&= \Delta\left(d_3(n)\Delta\left(\frac{d_2(n)a_2(n)A_2(n)\Delta(a_1(n)\Delta z(n)) + d_2(n)a_1(n)\Delta z(n)}{A_2(n)A_2(n+1)a_2(n)}\right)\right)\\
&= \Delta(d_3(n)(A_2(n+1)\Delta(a_2(n)\Delta(a_1(n)\Delta z(n))) - \Delta(a_1(n)\Delta z(n))) + \Delta(a_1(n)\Delta z(n)))\\
&= \Delta(a_3(n)\Delta(a_2(n)\Delta(a_1(n)\Delta z(n)))) = D_4z(n).
\end{aligned} \tag{51}$$

Clearly,

$$\sum_{n=n_0}^{\infty} \frac{1}{a_2(n)A_2(n)A_2(n+1)} = \sum_{n=n_0}^{\infty} \Delta\left(\frac{1}{A_2(n)}\right) = \lim_{n\to\infty} \frac{1}{A_2(n)} - \frac{1}{A_2(n_0)} = \infty, \tag{52}$$

so together with Equation (49), this shows that the operator is in canonical form. This completes the proof of the theorem. □

From Theorem 4, we see that under condition (49), Equation (1) can be written in the equivalent canonical form as follows:

$$\Delta(d_3(n)\Delta(d_2(n)\Delta(d_1(n)\Delta z(n)))) + q(n)x^\alpha(n-\tau) = 0. \tag{53}$$

That is, we have the following result that is analogous to Corollary 1.

Corollary 4. *Assume that condition (49) holds. Then $x(n)$ is a solution of Equation (1) if and only if it is a solution of (53).*

Using Equation (21) in (53) gives:

$$L_4z(n) + q(n)(1-p(n-\tau))^\alpha z^\alpha(n-\tau-\sigma) \leq 0, \tag{54}$$

where $L_0z(n)=z(n)$, $L_iz(n)=d_i(n)\Delta(L_{i-1}z(n))$, $i=1,2,3$, and $L_4z(n)=\Delta(L_3z(n))$.

Similar to what we did previously, we will now apply Theorem 2, Theorem 3, Corollary 2, and Corollary 3 to Equation (54) to obtain the following oscillation results.

To accomplish this, we need the additional notation:

$$Q_3(n)=\left(\frac{1}{d_2(n)}\sum_{s=n}^{\infty}\frac{1}{d_3(s)}\sum_{u=s}^{\infty}q(u)(1-p(u-\tau))^{\alpha}\right)\left(\sum_{s=n_*}^{n-\tau-\sigma-1}\frac{1}{d_1(s)}\right)^{\alpha}, \tag{55}$$

and

$$Q_4(n)=q(n)(1-p(n-\tau))^{\alpha}\left(\sum_{s=n_*}^{n-\sigma-\tau-1}\frac{1}{d_1(s)}\sum_{u=s_*}^{s-1}\frac{1}{d_2(u)}\sum_{s_1=n_*}^{u-1}\frac{1}{d_1(s_1)}\right)^{\alpha}, \tag{56}$$

where $n_*\geq n_0$ is a sufficiently large integer.

Theorem 5. *Let Equation (49) hold. If*

$$\sum_{n=n_1}^{\infty}q(n)(1-p(n-\tau))^{\alpha}=\infty, \tag{57}$$

then Equation (1) is oscillatory.

Theorem 6. *Let Equation (49) hold. Assume that, both of the first-order delay difference equations:*

$$\Delta w(n)+Q_3(n)w^{\alpha}(n-\sigma-\tau)=0, \tag{58}$$

and

$$\Delta w(n)+Q_4(n)w^{\alpha}(n-\sigma-\tau)=0, \tag{59}$$

are oscillatory. Then Equation (1) is oscillatory.

Corollary 5. *Assume that, Equation (49) holds. If*

$$\liminf_{n\to\infty}\sum_{s=n-\tau-\sigma}^{n-1}H_1(s)\geq\left(\frac{\tau+\sigma}{\tau+\sigma+1}\right)^{\tau+\sigma+1}\text{ for }\alpha=1, \tag{60}$$

or

$$\sum_{n=n_*}^{\infty}H_1(n)=\infty\text{ for }0<\alpha<1, \tag{61}$$

where

$$H_1(n)=\min\{Q_3(n),Q_4(n)\}, \tag{62}$$

then Equation (1) is oscillatory.

Corollary 6. *Assume that, Equation (49) holds. If $\alpha>1$ and there exists a constant $\lambda>\frac{1}{\tau+\sigma}\ln\alpha$ such that:*

$$\liminf_{n\to\infty}\left[H_1(n)\exp\left(-e^{\lambda n}\right)\right]>0, \tag{63}$$

where $H_1(n)$ is as in Corollary 5, then Equation (1) is oscillatory.

3. Examples

In this section, we present examples to illustrate our main results.

Example 1. Consider the semi-canonical fourth-order neutral delay difference equation as follows:

$$\Delta\left(n(n+1)\Delta\left(\frac{1}{n}\Delta^2z(n)\right)\right)+\frac{q_0}{n(n+2)}x(n-1)=0,\ n\geq1, \tag{64}$$

where $q_0>0$ and $z(n)=x(n)+\frac{1}{2}x(n-1)$. Here, we have $a_1(n)=1$, $a_2(n)=\frac{1}{n}$, $a_3(n)=n(n+1)$, $p(n)=\frac{1}{2}$, $\sigma=1$, $\tau=1$, $\alpha=1$, and $q(n)=q_0/(n(n+2))$. We easily see that (H_1)–(H_3) and Equation (6) hold, and $A_3(n)=\frac{1}{n}$, $b_1(n)=1$, $b_2(n)=1$, $b_3(n)=1$, and $Q(n)=q_0/(n(n+1)(n+2))$. The transformed equation then becomes (see (20))

$$\Delta^4z(n)+\frac{q_0}{n(n+1)(n+2)}x(n-1)=0, n\geq1, \tag{65}$$

which is clearly in canonical form. Further computations show that:

$$Q_1(n)\approx\frac{q_0(n-2)}{4n}, Q_2(n)\approx\frac{q_0(n-1)(n-2)(n-3)}{n(n+1)(n+2)}, \tag{66}$$

and

$$H(n)\approx\frac{q_0}{12}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n+1}\right)\left(1-\frac{3}{n+2}\right). \tag{67}$$

Condition (14) clearly holds and condition (44) becomes

$$\liminf_{n\to\infty}\sum_{s=n-2}^{n-1}\frac{q_0}{12}\left(1-\frac{1}{s}\right)\left(1-\frac{2}{s+1}\right)\left(1-\frac{3}{s+2}\right)=\frac{q_0}{12}>\left(\frac{2}{3}\right)^3, \tag{68}$$

that is, condition (44) holds if $q_0>\frac{32}{9}$. Therefore, by Corollary 2, the Equation (64) is oscillatory.

Example 2. Consider the fourth-order semi-canonical neutral delay difference equation:

$$\Delta\left(\frac{1}{n}\Delta\left(n(n+1)\Delta\left(\frac{1}{n}\Delta z(n)\right)\right)\right)+\frac{q_0}{n(n+1)(n+2)}x(n-1)$$
$$=0,\ n\geq 1, \tag{69}$$

where $q_0>0$ and $z(n)=x(n)+\frac{1}{2}x(n-1)$. We have $a_1(n)=\frac{1}{n}$, $a_2(n)=n(n+1)$, $a_3(n)=\frac{1}{n}$, $p(n)=\frac{1}{2}$, $\sigma=1$, $\tau=1$, $\alpha=1$, and $q(n)=q_0/(n(n+1)(n+2))$. It is easy to see that (H_1)–(H_3) and Equation (7) hold, and $A_2(n)=\frac{1}{n}$, and $d_1(n)=d_2(n)=d_3(n)=1$. The transformed equation becomes (see (53))

$$\Delta^4 z(n)+\frac{q_0}{n(n+1)(n+2)}x(n-1)=0, n\geq 1, \tag{70}$$

which is in canonical form. We see that Equation (49) holds, and further computations show that:

$$Q_3(n)\approx\frac{q_0(n-2)}{4n},\ Q_4(n)\approx\frac{q_0(n-1)(n-2)(n-3)}{n(n+1)(n+2)}, \tag{71}$$

and

$$H_1(n)\approx\frac{q_0}{12}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n+1}\right)\left(1-\frac{3}{n+2}\right). \tag{72}$$

Condition (60) becomes:

$$\liminf_{n\to\infty}\sum_{s=n-2}^{n-1}\frac{q_0}{12}\left(1-\frac{1}{s}\right)\left(1-\frac{2}{s+1}\right)\left(1-\frac{3}{s+2}\right)=\frac{q_0}{12}>\left(\frac{2}{3}\right)^3, \tag{73}$$

which holds if $q_0>\frac{32}{9}$. Therefore, by Corollary 5, Equation (69) is oscillatory.

Example 3. Consider the nonlinear semi-canonical neutral equation:

$$\Delta(n(n+1)\Delta^3 z(n))+q_0x^3(n-1)=0,\ n\geq 1, \tag{74}$$

where $z(n)=x(n)+\frac{1}{2}x(n-1)$ and $q_0>0$. In this case we have $a_1(n)=1$, $a_2(n)=1$, $a_3(n)=n(n+1)$, $p(n)=\frac{1}{2}$, $\sigma=1$, $\tau=1$, $\alpha=3$, and $q(n)=q_0$. We easily see that (H_1)–(H_3) and Equation (6) hold, $A_3(n)=\frac{1}{n}$, $b_1(n)=1$, $b_2(n)=n$, $b_3(n)=1$, and $Q(n)=\frac{q_0}{n+1}$. Equation (74) is transformed into the canonical equation

$$\Delta^2(n\Delta^2 z(n))+\frac{q_0}{n+1}x^3(n-1)=0. \tag{75}$$

Condition (25) becomes

$$\sum_{n=1}^{\infty}\frac{q_0}{8(n+1)}=\infty, \tag{76}$$

so it is satisfied as is condition (14). Therefore, by Theorem 2, the Equation (74) is oscillatory.

Example 4. Consider the equation:

$$\Delta^2(n(n+1)\Delta^2 z(n))+\frac{q_0}{n+1}x^{\frac{1}{3}}(n-1)=0,\ n\geq 1, \tag{77}$$

where $z(n)=x(n)+\frac{1}{2}x(n-1)$ and $q_0>0$. Here, $a_1(n)=1$, $a_2(n)=n(n+1)$, $a_3(n)=1$, $p(n)=\frac{1}{2}$, $\sigma=1$, $\tau=1$, $\alpha=\frac{1}{3}$, $q(n)=q_0/(n+1)$, $A_2(n)=\frac{1}{n}$, $d_1(n)=n$, $d_2(n)=1$, and $d_3(n)=n$. Conditions (H_1)–(H_3) and Equation (7) hold. The transformed equation is as follows:

$$\Delta(n\Delta^2(n\Delta z(n)))+\frac{q_0}{n+1}x^{\frac{1}{3}}(n-1)=0, \tag{78}$$

which is in canonical form. Condition (49) holds and Equation (61) takes the form:

$$\sum_{n=1}^{\infty}\frac{1}{2^{\frac{1}{3}}(n+1)}=\infty, \tag{79}$$

which is clearly satisfied. Therefore, by Corollary 5, Equation (77) is oscillatory.

In our final two examples, we are able to actually exhibit an oscillatory solution.

Example 5. Consider the neutral equation:

$$\Delta\left(n(n+1)\Delta\left(\frac{1}{n}\Delta^2 z(n)\right)\right)+\frac{8(n^2+4n+3)}{n-1}x(n-1)=0, n\geq 2. \tag{80}$$

Here, we have $a_1(n)=1$, $a_2(n)=\frac{1}{n}$, $a_3(n)=n(n+1)$, $p(n)=\frac{1}{2}$, $\tau=\sigma=1$, $\alpha=1$, and $q(n)=8(n^2+4n+3)/(n-1)$. Conditions (H_1)–(H_4) hold. Also, $A_3(n)=\frac{1}{n}$, $b_1(n)=b_2(n)=b_3(n)=1$, and $Q(n)=4(n+3)/(n-1)$. The transformed equation is as follows:

$$\Delta^4 z(n)+\frac{4(n+3)}{n-1}x(n-1)=0, n\geq 2, \tag{81}$$

which we see is in canonical form. Condition (14) clearly holds and condition (25) becomes:

$$\sum_{n=2}^{\infty} \frac{4(n+3)}{n-1} = \infty. \tag{82}$$

By Theorem 2, Equation (80) is oscillatory, and the sequence $x(n) = \{n(-1)^n\}$ is one such oscillatory solution.

Example 6. Consider the nonneutral equation

$$\Delta\left(n(n+1)\Delta\left(\frac{1}{n}\Delta^2 x(n)\right)\right) + \frac{16(n+1)(n+2)}{n-1}x(n-1) = 0, \ n \geq 1, \tag{83}$$

where we have $a_1(n) = 1$, $a_2(n) = \frac{1}{n}$, $a_3(n) = n(n+1)$, $p(n) \equiv 0$, $\tau = 1$, $\alpha = 1$, and $q(n) = 16(n+1)(n+2)/(n-1)$. Conditions (H_1)–(H_4) hold. Also, $A_3(n) = \frac{1}{n}$, $b_1(n) = 1$, $b_2(n) = 1$, $b_3(n) = 1$, and $Q(n) = 16(n+2)/(n-1)$, so the transformed equation is as follows:

$$\Delta^4 z(n) + \frac{16(n+2)}{n-1}x(n-1) = 0, n \geq 2, \tag{84}$$

which we see is in canonical form. Condition (14) clearly holds. Condition (25) becomes:

$$\sum_{n=2}^{\infty} \frac{16(n+2)}{n-1} = \infty. \tag{85}$$

By Theorem 2, Equation (83) is oscillatory, and the sequence $z(n) = x(n) = \{n(-1)^n\}$ is an oscillatory solution.

4. Conclusion

In this paper, we introduced a technique to convert the semi-canonical Equation (1) with (6) or (7) holding into a canonical type equation. This simplified the investigation of the oscillatory properties of the solutions. Examples are presented to illustrate the importance of the main results. One significant improvement over the papers [4, 8, 10, 15, 16, 21–23] is that we only need to eliminate two types of nonoscillatory solutions to obtain our oscillation criteria. We leave the study of Equation (1) in the cases where (8), (9), (10), or (–11) holds to the future research efforts.

Data Availability

No data sets were created during the conduction of this research.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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