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A Study on the Norms of Toeplitz Matrices with the Generalized Mersenne Numbers

Sevda Aktaş [a*] **and Yüksel Soykan** [a]

[a] *Department of Mathematics, Art and Science Faculty, Zonguldak Bülent Ecevit University, 67100, Zonguldak, Turkey.*

Authors' contributions

This work was carried out in collaboration between both authors. Both authors have read and approved the final version of the manuscript.

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ABSTRACT

In this article presents findings related to Toeplitz matrices with Mersenne numbers. We started by creating Toeplitz matrices whose components are Mersenne numbers. We then calculated the Euclidian, row and column norms of these matrices and established both lower and upper bounds for their spectral norms. Additionally, we determined the upper bounds for the Frobenius and spectral norms of the Kronecker and Hadamard product matrices of the Toeplitz matrices with Mersenne numbers.

Keywords: Mersenne numbers; Toeplitz matrix; norm; Hadamard product; Kronecker product.

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**Corresponding author: E-mail: sevdamamuraktas@gmail.com;*

1 INTRODUCTION

Special matrices, with a particular emphasis on Toeplitz matrices, have been the subject of extensive scholarly publications. These matrices exhibit unique number sequences, including Fibonacci, Lucas, k-Fibonacci, k-Lucas, Pell, Pell-Lucas, modified Pell, Jacobsthal, Jacobsthal-Lucas and k-Jacobsthal numbers. In these articles, researchers have focused their efforts on conducting thorough research on the Frobenius norms of Toeplitz matrices. They have notably highlighted the numerical characteristics derived from these distinct sequences and aimed to establish both lower and upper bounds for the spectral norms of these matrices, thereby enhancing our understanding of their inherent properties. This field has benefited from contributions made by numerous researchers. For instance, Solak [1] dedicated their research to calculating the spectral norms of Toeplitz matrices associated with Fibonacci and Lucas numbers. Akbulak and Bozkurt [2] successfully derived specialized norms for Toeplitz matrices characterized by Fibonacci and Lucas numbers, enhancing their contributions by establishing bounds for the spectral norm. Shen [3], in their work, extended these insights to Toeplitz matrices featuring k-Fibonacci and k-Lucas numbers, not only presenting specific norms but also defining bounds for the spectral norms. Shen and their colleagues [3] further explored the lower and upper bounds for spectral norms, expanding their investigation to include Frobenius norms, with a particular focus on Hadamard and Kronecker products involving these matrices. Eylem G. Karpuz's meticulous work deserves special recognition [4] for its comprehensive examination of the norms of Toeplitz matrices with elements represented by Pell numbers. Daşdemir [5] also made noteworthy contributions to this field by investigating the special norms of Toeplitz matrices, including those related to Pell, Pell-Lucas, and modified Pell numbers. Daşdemir and their colleagues added greater depth to the field by deriving comprehensive lower and upper bounds for the spectral norm. Uygun's contributions are equally worthy of attention. In their 2019 work, Uygun [6] focused on Toeplitz matrices with Jacobsthal and Jacobsthal-Lucas numbers, yielding special norms and establishing both lower and upper bounds for the spectral norm. Moreover, Uygun's research extended to determining the upper bound of the Frobenius norm for the Kronecker and Hadamard products of these matrices, further advancing our understanding of these mathematical structures. Uygun's parallel study on the k-Jacobsthal and k-Jacobsthal-Lucas numbers adds yet another layer of depth to this area of research [7]. The efforts of these researchers have substantially expanded our knowledge of special matrices shedding light on their properties and paving the way for further exploration in this field.

First, we present some information on generalized Mersenne numbers.

A generalized Mersenne sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relation

$$W_n = 3W_{n-1} - 2W_{n-2} \tag{1.1}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = \frac{3}{2}W_{-(n-1)} - \frac{1}{2}W_{-(n-2)}$$

for $n = 1, 2, 3, \cdots$. Therefore, recurrence equation (1.1) holds for all integer n.

The first few generalized Mersenne numbers with positive subscript and negative subscript are given in the following Table 1.

For more information on generalized Mersenne numbers, see for example, Soykan [8].

Mersenne sequence $\{M_n\}_{n\geq 0}$ and Mersenne-Lucas sequence $\{H_n\}_{n\geq 0}$ are defined respectively, by the second order recurrence relations;

$$\begin{aligned} M_n &= 3M_{n-1} - 2M_{n-2} \quad M_0 = 0, M_1 = 1, &(1.2)\\ H_n &= 3H_{n-1} - 2H_{n-2}, \quad H_0 = 2, H_1 = 3 &(1.3)\end{aligned}$$

The sequences $\{M_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} M_{-n} &= \frac{3}{2}M_{-(n-1)} - \frac{1}{2}M_{-(n-2)},\\ H_{-n} &= \frac{3}{2}H_{-(n-1)} - \frac{1}{2}H_{-(n-2)}.\end{aligned}$$

for $n = 1, 2, 3, \cdots$ respectively.

Therefore recurrence equation (1.2), equation (1.3) hold for all integer n.

Next, we present the first few values of the Mersenne and Mersenne-Lucas numbers with positive and negative subscripts:

Table 1. A few generalized Mersenne numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$\frac{3}{2}W_0 - \frac{1}{2}W_1$
2	$3W_1 - 2W_0$	$\frac{7}{4}W_0 - \frac{3}{4}W_1$
3	$7W_1 - 6W_0$	$\frac{15}{8}W_0 - \frac{7}{8}W_1$
4	$15W_1 - 14W_0$	$\frac{31}{16}W_0 - \frac{15}{16}W_1$
5	$31W_1 - 30W_0$	$\frac{63}{32}W_0 - \frac{31}{32}W_1$
6	$63W_1 - 62W_0$	$\frac{127}{64}W_0 - \frac{63}{64}W_1$
7	$127W_1 - 126W_0$	$\frac{255}{128}W_0 - \frac{127}{128}W_1$
8	$255W_1 - 254W_0$	$\frac{511}{256}W_0 - \frac{255}{256}W_1$
9	$511W_1 - 510W_0$	$\frac{1023}{512}W_0 - \frac{511}{512}W_1$
10	$1023W_1 - 1022W_0$	$\frac{2047}{1024}W_0 - \frac{1023}{1024}W_1$
11	$2047W_1 - 2046W_0$	$\frac{4095}{2048}W_0 - \frac{2047}{2048}W_1$
12	$4095W_1 - 4094W_0$	$\frac{8191}{4096}W_0 - \frac{4095}{4096}W_1$

Table 2. The first few values of the special second-order numbers with positive and negative subscripts

n	0	1	2	3	4	5	6	7	8	9	10	11	12
M_n	0	1	3	7	15	31	63	127	255	511	1023	2047	4095
M_{-n}	0	$-\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{7}{8}$	$-\frac{15}{16}$	$-\frac{31}{32}$	$-\frac{63}{64}$	$-\frac{127}{128}$	$-\frac{255}{256}$	$-\frac{511}{512}$	$-\frac{1023}{1024}$	$-\frac{2047}{2048}$	$-\frac{4095}{4096}$
H_n	2	3	5	9	17	33	65	129	257	513	1025	2049	4097
H_{-n}	2	$\frac{3}{2}$	$\frac{5}{4}$	$\frac{9}{8}$	$\frac{17}{16}$	$\frac{33}{32}$	$\frac{65}{64}$	$\frac{129}{128}$	$\frac{257}{256}$	$\frac{513}{512}$	$\frac{1025}{1024}$	$\frac{2049}{2048}$	$\frac{4097}{4096}$

Characteristic equation of generalized Mersenne sequence $\{W_n\}_{n\geq 0}$ is given as the quadratic equation

$$x^2 - 3x + 2 = 0,$$

whose roots are α, β and

$$\begin{aligned} \alpha &= 2 \\ \beta &= 1. \end{aligned}$$

Binet's formula of Generalized Mersenne sequence is given as

$$\begin{aligned} W_n &= \frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n \\ &= (W_1 - W_0)2^n - (W_1 - 2W_0). \end{aligned}$$

Binet's formulas of Mersenne and Mersenne-Lucas are

$$\begin{aligned} M_n &= \frac{\alpha^n}{(\alpha - \beta)} + \frac{\beta^n}{(\beta - \alpha)} = 2^n - 1, \\ H_n &= \alpha^n + \beta^n = 2^n + 1 \end{aligned}$$

and Binet's formulas of Mersenne numbers and Mersenne-Lucas at the negative index are

$$\begin{aligned} M_{-n} &= \frac{1}{\alpha^n} - \frac{1}{\beta^n} = \frac{-2^n + 1}{2^n}, \\ H_{-n} &= \frac{1}{\alpha^n} + \frac{1}{\beta^n} = \frac{2^n + 1}{2^n}. \end{aligned}$$

2 PRELIMINARIES

A matrix $T=[t_{ij}]\in M_n(\mathbb{C})$ is called a Toeplitz matrix if it is of the form $t_{ij}=t_{i-j}$ for

$$T_n=\begin{pmatrix} t_0 & t_{-1} & t_{-2} & \cdots & t_{1-n} \\ t_1 & t_0 & t_{-1} & \cdots & t_{2-n} \\ t_2 & t_1 & t_0 & \cdots & t_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_0 \end{pmatrix}.$$

Now, we give some preliminaries related to our study. Let $A=(a_{ij})$ be an $m\times n$ matrix. The ℓ_p norm of the matrix A is defined by

$$\|A\|_p=(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^p)^{\frac{1}{p}} \qquad (1\leq p<\infty).$$

If $p=\infty$, then $\|A\|_\infty=\lim_{p\to\infty}\|A\|_p=\max_{i,j}|a_{ij}|$.

The well-known Frobenius (Euclidean) and spectral norms of the matrix A are defined respectively by

$$\|A\|_F=(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2)^{\frac{1}{2}}$$

and

$$\|A\|_2=\sqrt{\max_{1\leq i\leq n}|\lambda_i|} \tag{2.1}$$

where the numbers λ_i are the eigenvalues of matrix A^HA and the matrix A^H is the conjugate transpose of the matrix A. The following inequality between the Frobenius and spectral norms of A holds.

$$\frac{1}{\sqrt{n}}\|A\|_F\leq\|A\|_2\leq\|A\|_F. \tag{2.2}$$

It follows that

$$\|A\|_2\leq\|A\|_F\leq\sqrt{n}\|A\|_2.$$

In literature, there are other types of norms of matrices. The maximum column sum matrix norm of $n\times n$ matrix $A=(a_{ij})$ is

$$\|A\|_1=\max_{1\leq j\leq n}\sum_{i=1}^{n}|a_{ij}| \tag{2.3}$$

and the maximum row sum matrix norm is

$$\|A\|_\infty=\max_{1\leq i\leq n}\sum_{j=1}^{n}|a_{ij}|. \tag{2.4}$$

The maximum column lenght norm $c_1(.)$ and maximum row lenght norm $r_1(.)$ of on matrix of order $m\times n$ are defined as follows

$$c_1(A)\equiv\max_{1\leq j\leq n}\left(\sum_{i=1}^{m}|a_{ij}|^2\right)^{\frac{1}{2}}=\max_{1\leq j\leq n}\left\|[a_{ij}]_{i=1}^{m}\right\|_F \tag{2.5}$$

and

$$r_1(A)\equiv\max_{1\leq i\leq m}\left(\sum_{j=1}^{n}|a_{ij}|^2\right)^{\frac{1}{2}}=\max_{1\leq i\leq m}\left\|[a_{ij}]_{j=1}^{n}\right\|_F \tag{2.6}$$

respectively.

For any $A,B\in M_{mn}(\mathbb{C})$, the Hadamard product of $A=(a_{ij})$ and $B=(b_{ij})$ is entrywise product and defined by $A\circ B=(a_{ij}b_{ij})$ and have the following properties

$$\left\Vert A\circ B\right\Vert _{2}\leq r_{1}\left(A\right) c_{1}\left(B\right) , \tag{2.7}$$

and

$$\left\Vert A\circ B\right\Vert _{2}\leq \left\Vert A\right\Vert _{2}\left\Vert B\right\Vert _{2}. \tag{2.8}$$

In addition,

$$\left\Vert A\circ B\right\Vert _{F}\leq \left\Vert A\right\Vert _{F}\left\Vert B\right\Vert _{F}. \tag{2.9}$$

Let $A\in M_{mn}(\mathbb{C})$, and $B\in M_{mn}(\mathbb{C})$ be given, then the Kronecker product of A,B is defined by

$$\left\Vert A\otimes B\right\Vert =\begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}$$

and have the following properties:

$$\begin{aligned} \left\Vert A\otimes B\right\Vert _{2} &= \left\Vert A\right\Vert _{2}\left\Vert B\right\Vert _{2}, \\ \left\Vert A\otimes B\right\Vert _{F} &= \left\Vert A\right\Vert _{F}\left\Vert B\right\Vert _{F}. \end{aligned} \tag{2.10}$$

In the following theorem, we give some formulas of generalized Mersenne of numbers.

Theorem 1. *For generalized Mersenne numbers, we have following sum formulas:*

(a) *[8, Proposition 22. a] If* $2x^2-3x+1=0,\ i.e.,\ x=1$ *or* $x=\frac{1}{2}$, *then*

$$\sum_{k=0}^{n}x^{k}W_{k}=\frac{(2(n+2)x-3(n+1))x^{n}W_{n}+2(n+1)x^{n}W_{n-1}+(W_{1}-3W_{0})}{4x-3}.$$

(b) *[9, Proposition 2.1. a] If* $(2x-1)(4x-1)(x-1)=0,\ i.e.,\ x=\frac{1}{4}$ *or* $x=\frac{1}{2}$ *or* $x=1$ *then*

$$\sum_{k=0}^{n}x^{k}W_{k}^{2}=\frac{\Psi}{(24x^{2}-28x+7)}$$

where
$\Psi=(n(2x-1)(4x-5)+24x^{2}-28x+5)x^{n}W_{n}^{2}+4((2x-1)n+4x-1)x^{n}W_{n-1}^{2}+2W_{0}^{2}-(4x-1)(3W_{0}-W_{1})^{2}+2(W_{1}^{2}+2W_{0}^{2}-3W_{1}W_{0})(2^{n}(n+1)x^{n}-1).$

(c) *[9, Proposition 2.1. d] If* $(x-2)(x-1)(x-4)=0,\ i.e.,\ x=1$ *or* $x=2$ *or* $x=4$ *then*

$$\sum_{k=0}^{n}x^{k}W_{-k}^{2}=\frac{\Psi}{(3x^{2}-14x+14)}$$

where

$\Psi=(n(x-2)+2(x-1))x^{n}W_{-n+1}^{2}+(n(x-2)(x-5)+3x^{2}-14x+10)x^{n}W_{-n}^{2}+4W_{0}^{2}-2(x-1)W_{1}^{2}+4(W_{1}^{2}+2W_{0}^{2}-3W_{1}W_{0})(2^{-n}(n+1)x^{n}-1).$

If we set $x=1$ in the last Theorem, we have the following corollary.

Corollary 2. *For generalized Mersenne numbers, we have following sum formulas:*

(a)

$$\sum_{k=0}^{n}W_{k}=(1-n)W_{n}+(2n+2)W_{n-1}+(W_{1}-3W_{0}). \tag{2.11}$$

(b)

$$\sum_{k=0}^{n} W_k^2 = \frac{1}{3}((1-n)W_n^2+4(n+3)W_{n-1}^2-25W_0^2+18W_0W_1-3W_1^2+2(W_1-2W_0)(W_1-W_0)(2^n(n+1)-1)). \tag{2.12}$$

(c)

$$\sum_{k=0}^{n} W_{-k}^2 = \frac{1}{3}(-nW_{-n+1}^2+(4n-1)W_{-n}^2+4W_0^2+4(W_1^2+2W_0^2-3W_1W_0)(2^{-n}(n+1)-1)). \tag{2.13}$$

3 MAIN RESULTS

In this paper we use the notation $A = T(W_0, W_1, \cdots, W_{n-1})$ for the Toeplitz matrix with generalized Mersenne numbers, i.e.,

$$A = \begin{pmatrix} W_0 & W_{-1} & W_{-2} & \cdots & W_{1-n} \\ W_1 & W_0 & W_{-1} & \cdots & W_{2-n} \\ W_2 & W_1 & W_0 & \cdots & W_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_{n-1} & W_{n-2} & W_{n-3} & \cdots & W_0 \end{pmatrix}. \tag{3.1}$$

For exclusive cases, we get

$$A = \begin{pmatrix} M_0 & M_{-1} & M_{-2} & \cdots & M_{1-n} \\ M_1 & M_0 & M_{-1} & \cdots & M_{2-n} \\ M_2 & M_1 & M_0 & \cdots & M_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{n-1} & M_{n-2} & M_{n-3} & \cdots & M_0 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{3}{4} & \cdots & M_{1-n} \\ 1 & 0 & -\frac{1}{2} & \cdots & M_{2-n} \\ 3 & 1 & 0 & \cdots & M_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{n-1} & M_{n-2} & M_{n-3} & \cdots & 0 \end{pmatrix} \tag{3.2}$$

for the Toeplitz matrix $A = T(M_0, M_1, \cdots, M_{n-1})$ with Mersenne numbers and

$$A = \begin{pmatrix} H_0 & H_{-1} & H_{-2} & \cdots & H_{1-n} \\ H_1 & H_0 & H_{-1} & \cdots & H_{2-n} \\ H_2 & H_1 & H_0 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & H_{n-2} & H_{n-3} & \cdots & H_0 \end{pmatrix} = \begin{pmatrix} 2 & \frac{3}{2} & \frac{5}{4} & \cdots & H_{1-n} \\ 3 & 2 & \frac{3}{2} & \cdots & H_{2-n} \\ 5 & 3 & 2 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & H_{n-2} & H_{n-3} & \cdots & 2 \end{pmatrix} \tag{3.3}$$

for the Toeplitz matrix $A = T(H_0, H_1, \cdots, H_{n-1})$ with Mersenne- Lucas numbers.

In the following theorem, we present the norm value of $\|A\|_1$ and $\|A\|_\infty$ of the largest absolute column sum and the largest absolute row sum of A.

Theorem 3. *Let $A = T(W_0, W_1, \cdots, W_{n-1})$ be a Toeplitz matrix with generalized Mersenne numbers then the largest absolute column sum (1-norm) and the largest absolute row sum (∞-norm) of A are*

$$\|A\|_1 = \|A\|_\infty = \begin{cases} nW_n-(2n+2)W_{n-1}+3W_0-W_1 & , \text{ if } |W_k| \geq |W_{-k}| \text{ and } W_k \leq 0 \\ -nW_n+(2n+2)W_{n-1}+(W_1-3W_0) & , \text{ if } |W_k| \geq |W_{-k}| \text{ and } W_k \geq 0 \end{cases}$$

where $k = i - j : i, j = 0, 1, \cdots, n-1; k \in N, -k \in N^-$.

Proof**.** Acknowledge $A = T(W_0, W_1, \cdots, W_{n-1})$ which is given as in (3.1). By the definitions of $1 - norm$ and $\infty - norm$, and equation (2.3), equation (2.4) and equation(2.11), we conclude that

(i) If $|W_k| \geq |W_{-k}|$, $k \in N$ and $W_k \leq 0, k \in N$, then we get

$$\begin{aligned}
\|A\|_1 &= \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}| = \max\{|a_{1j}| + |a_{2j}| + |a_{3j}| + \cdots + |a_{nj}|\} = \sum_{i=1}^{n} |a_{i1}| \\
&= |a_{11}| + |a_{21}| + |a_{31}| + \cdots + |a_{n1}| = \sum_{k=0}^{n-1} |W_k| \\
&= -(\sum_{k=0}^{n-1} W_k) = -(\sum_{k=0}^{n} W_k - W_n) = -\sum_{k=0}^{n} W_k + W_n \\
&= -((1-n)W_n + (2n+2)W_{n-1} + (W_1 - 3W_0)) + W_n \\
&= (n-1)W_n - (2n+2)W_{n-1} + (3W_0 - W_1) + W_n \\
&= nW_n - (2n+2)W_{n-1} + 3W_0 - W_1
\end{aligned}$$

and if $|W_k| \geq |W_{-k}|$, $k \in N$ and $W_k \geq 0, k \in N$, then we obtain

$$\begin{aligned}
\|A\|_1 &= \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{ij}| = \max\{|a_{1j}| + |a_{2j}| + |a_{3j}| + \cdots + |a_{nj}|\} = \sum_{i=1}^{n} |a_{i1}| \\
&= |a_{11}| + |a_{21}| + |a_{31}| + \cdots + |a_{n1}| = \sum_{k=0}^{n-1} |W_k| \\
&= \sum_{k=0}^{n-1} W_k = \sum_{k=0}^{n} W_k - W_n \\
&= (1-n)W_n + (2n+2)W_{n-1} + (W_1 - 3W_0) - W_n \\
&= -nW_n + (2n+2)W_{n-1} + (W_1 - 3W_0).
\end{aligned}$$

(ii) If $|W_k| \geq |W_{-k}|$, $k \in N$ and $W_k \leq 0, k \in N$, then it follows that

$$\begin{aligned}
\|A\|_\infty &= \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}| = \max\{|a_{i1}| + |a_{i2}| + |a_{i3}| + \cdots + |a_{in}|\} = \sum_{j=1}^{n} |a_{nj}| \\
&= |a_{n1}| + |a_{n2}| + |a_{n3}| + \cdots + |a_{nn}| = \sum_{k=0}^{n-1} |W_k| \\
&= -(\sum_{k=0}^{n} W_k - W_n) = -\sum_{k=0}^{n} W_k + W_n \\
&= -((1-n)W_n + (2n+2)W_{n-1} + (W_1 - 3W_0)) + W_n \\
&= nW_n - (2n+2)W_{n-1} + 3W_0 - W_1
\end{aligned}$$

and if $|W_{-k}| \geq |W_k|$, $k \in N$ and $W_k \geq 0, k \in N$, then we get

$$\begin{aligned}
\|A\|_\infty &= \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{ij}| = \max\{|a_{i1}| + |a_{i2}| + |a_{i3}| + \cdots + |a_{in}|\} = \sum_{j=1}^{n} |a_{nj}| \\
&= |a_{n1}| + |a_{n2}| + |a_{n3}| + \cdots + |a_{nn}| = \sum_{k=0}^{n-1} |W_k| = \sum_{k=0}^{n} W_k - W_n \\
&= (1-n)W_n + (2n+2)W_{n-1} + (W_1 - 3W_0) - W_n \\
&= -nW_n + (2n+2)W_{n-1} + (W_1 - 3W_0).
\end{aligned}$$

Thus, the proof is completed. □

Remark 4. *In the statement of the Theorem 3 the condion on W_n, W_{-n}, $n \in N$ is given to calculade $\|A\|_1$ and $\|A\|_\infty$ norms of Mersenne, Mersenne-Lucas numbers. The other cases can be handled similarly.*

From the last Theorem 3, we have the following corollary which gives norm value of $\|A\|_1$ and $\|A\|_\infty$ of the largest absolute column sum and the largest absolute row sum of A with Mersenne numbers and Mersenne-Lucas numbers, respectively, (set $W_n = M_n$ with $M_0 = 0, M_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 3$, respectively).

Corollary 5.

(a) For $A = T(M_{0,}M_1, \cdots, M_{n-1})$, the vaules of norms of Toeplitz matrices with Mersenne numbers hold the following property:

$$\|A\|_1 = \|A\|_\infty = -nM_n + (2n+2)M_{n-1} + 1$$

(b) For $A = T\left(H_0, H_1, \cdots, H_{n-1}\right)$, the values of norms of Toeplitz matrices with Mersenne-Lucas numbers hold the following property:

$$\|A\|_1 = \|A\|_\infty = -nH_n + (2n+2)H_{n-1} - 3.$$

Next theorem presents the Frobenious (Euclidian) norm of a Toeplitz matrix A.

Theorem 6. *Consider $A = T\left(W_0, W_1, \cdots, W_{n-1}\right)$ which is given in (3.1), then the Frobenious (Euclidian) norm of matrix A is*

$$\|A\|_F = \sqrt{\Lambda_1}$$

where

$$\Lambda_1 = (\tfrac{-3n^2+5n}{18})W_{-n+1}^2+(\tfrac{6n^2-10n+4}{9})W_{-n}^2-(\tfrac{144n+271}{18})W_0^2+(6n+11)W_0W_1-\tfrac{(6n+11)}{6}W_1^2-(\tfrac{3n^2+23n+34}{18})W_n^2+\tfrac{6n^2+46n+84}{9}W_{n-1}^2-2W_{-1}^2+\tfrac{1}{9}(6n^2 2^{-n}+3n^2 2^n-10n2^{-n}+23n2^n-28(2^{-n})+14(2^n)-18n+14)(W_1-2W_0)(W_1-W_0).$$

Proof. The matrix A is of the form

$$A = \begin{pmatrix} W_0 & W_{-1} & W_{-2} & \cdots & W_{1-n} \\ W_1 & W_0 & W_{-1} & \cdots & W_{2-n} \\ W_2 & W_1 & W_0 & \cdots & W_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_{n-1} & W_{n-2} & W_{n-3} & \cdots & W_0 \end{pmatrix}.$$

Then we have

$$\begin{aligned} \|A\|_F^2 &= nW_0^2 + (n-1)W_{-1}^2 + (n-2)W_{-2}^2 + (n-3)W_{-3}^2 + \cdots + W_{1-n}^2 \\ &\quad +(n-1)W_1^2 + (n-2)W_2^2 + (n-3)W_3^2 + \cdots + W_{n-1}^2 \end{aligned}$$

and so

$$\begin{aligned}
\|A\|_F^2 &= nW_0^2+\sum_{k=1}^{n-1}(\sum_{i=0}^{k}W_i^2)+\sum_{k=1}^{n-1}(\sum_{i=0}^{k}W_{-i}^2)-2(n-1)W_0^2 \\
&= (n-2n+2)W_0^2+\sum_{k=1}^{n-1}(\sum_{i=0}^{k}W_i^2)+\sum_{k=1}^{n-1}(\sum_{i=0}^{k}W_{-i}^2) \\
&= (\frac{-3n^2+5n}{18})W_{-n+1}^2+(\frac{6n^2-10n+4}{9})W_{-n}^2-(\frac{144n+271}{18})W_0^2 \\
&\quad +(6n+11)W_0W_1-\frac{(6n+11)}{6}W_1^2-(\frac{3n^2+23n+34}{18})W_n^2+\frac{6n^2+46n+84}{9}W_{n-1}^2 \\
&\quad -2W_{-1}^2+\frac{1}{9}(6n^2 2^{-n}+3n^2 2^n-10n2^{-n}+23n2^n-28(2^{-n}) \\
&\quad +14(2^n)-18n+14)(W_1-2W_0)(W_1-W_0) \\
&= (\frac{-3n^2+5n}{18})W_{-n+1}^2+(\frac{6n^2-10n+4}{9})W_{-n}^2-(\frac{3n^2+23n+34}{18})W_n^2+\frac{6n^2+46n+84}{9}W_{n-1}^2 \\
&\quad +\frac{W_0^2}{18}((24n^2-40n-112)2^{-n}+(12n^2+92n+56)2^n-216n-215) \\
&\quad +\frac{W_0W_1}{3}((-6n^2+10n-28)2^{-n}-(3n^2+23n+14)2^n+36n+19) \\
&\quad +\frac{W_1^2}{18}((12n^2-20n-56)2^{-n}+(6n^2+46n+28)2^n-54n-5)-2W_{-1}^2
\end{aligned}$$

Moreover, we use equation 2.12 and equation 2.13 in Corollary 2.

Therefore, we get

$\|A\|_F^2=(\frac{-3n^2+5n}{18})W_{-n+1}^2+(\frac{6n^2-10n+4}{9})W_{-n}^2-(\frac{3n^2+23n+34}{18})W_n^2+\frac{6n^2+46n+84}{9}W_{n-1}^2-2W_{-1}^2+\frac{W_0^2}{18}((24n^2-40n-112)(2^{-n})+(12n^2+92n+56)2^n-216n-215)$
$+\frac{W_0W_1}{3}((-6n^2+10n+28)(2^{-n})-(3n^2+23n+14)2^n+36n+19)+\frac{W_1^2}{18}((12n^2-20n-56)(2^{-n})+(6n^2+46n+28)2^n-54n-5)$

This complates the proof. □

From the last Theorem 6, we have the following corollary which gives Frobenius norm formulas of Mersenne numbers and Mersenne-Lucas numbers, respectively, (take $W_n=M_n$ with $M_0=0, M_1=1, M_{-1}=-\frac{1}{2}$ and $W_n=H_n$ with $H_0=2, H_1=3, H_{-1}=\frac{3}{2}$, respectively).

Corollary 7. *For $n\geq 0$, Toeplitz matrices with the Mersenne and Mersenne-Lucas numbers, respectively have the following properties:*

(a) $\|A\|_F=\sqrt{\Lambda_2}$
where A is given as in (3.2)
$\Lambda_2=(\frac{-3n^2+5n}{18})M_{-n+1}^2+(\frac{6n^2-10n+4}{9})M_{-n}^2-(\frac{3n^2+23n+34}{18})M_n^2+\frac{6n^2+46n+84}{9}M_{n-1}^2+\frac{1}{9}(6n^2 2^{-n}+3n^2 2^n-10n2^{-n}+23n2^n-28(2^{-n})+14(2^n)-27n-7)$.

(b) $\|A\|_F=\sqrt{\Lambda_3}$
where A is given as in (3.3)
$\Lambda_3=(\frac{-3n^2+5n}{18})H_{-n+1}^2+(\frac{6n^2-10n+4}{9})H_{-n}^2-(\frac{3n^2+23n+34}{18})H_n^2+(\frac{6n^2+46n+84}{9})H_{n-1}^2-\frac{1}{9}(6n^2 2^{-n}+3n^2 2^n-10n2^{-n}+23n2^n-28(2^{-n})+14(2^n)+27n+151)$.

In the following theorem, we find the lower and upper bounds for the spectral norm of the matrices with the Mersenne numbers, Mersenne-Lucas numbers, respectively, (take $W_n=M_n$ with $M_0=0, M_1=1$ and $W_n=H_n$ with $H_0=2, H_1=3$, respectively).

Theorem 8.

(a) Consider $A = T(M_0, M_1, \cdots, M_{n-1})$ which is given as in (3.2). Let

$$C = \begin{pmatrix} 1 & M_{-1} & M_{-2} & \cdots & M_{1-n} \\ 1 & M_0 & M_{-1} & \cdots & M_{2-n} \\ 1 & M_1 & M_0 & \cdots & M_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & M_{n-2} & M_{n-3} & \cdots & M_0 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{3}{4} & \cdots & M_{1-n} \\ 1 & 0 & -\frac{1}{2} & \cdots & M_{2-n} \\ 1 & 1 & 0 & \cdots & M_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & M_{n-2} & M_{n-3} & \cdots & 0 \end{pmatrix}.$$

and

$$D = \begin{pmatrix} M_0 & 1 & 1 & \cdots & 1 \\ M_1 & 1 & 1 & \cdots & 1 \\ M_2 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 3 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix}$$

such that $A = C \circ D$ (Hadamard Product of C and D).

(i)

$$\|A\|_2 \geq \sqrt{\frac{1}{n}\Lambda_2}$$

where Λ_2 is as in Corollary 7.

(ii)

$$\|A\|_2 \leq \Lambda_4$$

where

$$\begin{aligned} \Lambda_4 &= \left(\frac{1}{3}((-2-n)M_n^2 + (4n+9)M_{n-1}^2 + 2^{n+1}(n+1) - 2)\right)^{\frac{1}{2}} \\ &\quad \times\left(\frac{1}{3}((-2-n)M_n^2 + 4(n+3)M_{n-1}^2 + 2^{n+1}(n+1) - 5)\right)^{\frac{1}{2}}. \end{aligned}$$

(b) Consider $A = T(H_0, H_1, \cdots, H_{n-1})$ which is given as in (3.3). Let

$$C = \begin{pmatrix} 1 & H_{-1} & H_{-2} & \cdots & H_{1-n} \\ 1 & H_0 & H_{-1} & \cdots & H_{2-n} \\ 1 & H_1 & H_0 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & H_{n-2} & H_{n-3} & \cdots & H_0 \end{pmatrix} = \begin{pmatrix} 1 & \frac{3}{2} & \frac{5}{4} & \cdots & H_{1-n} \\ 1 & 2 & \frac{3}{2} & \cdots & H_{2-n} \\ 1 & 3 & 2 & \cdots & H_{3-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & H_{n-2} & H_{n-3} & \cdots & 2 \end{pmatrix}.$$

and

$$D = \begin{pmatrix} H_0 & 1 & 1 & \cdots & 1 \\ H_1 & 1 & 1 & \cdots & 1 \\ H_2 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 3 & 1 & 1 & \cdots & 1 \\ 5 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_{n-1} & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

such that $A = C \circ D$ (Hadamard Product of C and D).

(i)

$$\|A\|_2 \geq \sqrt{\frac{1}{n}\Lambda_3}$$

where Λ_3 is as in Corollary 7

(ii)

$$\left\|A\right\|_2 \leq \Lambda_5$$

where

$$\begin{aligned}\Lambda_5 &= (\frac{1}{3}((-2-n)H_n^2+(4n+9)H_{n-1}^2-2^{n+1}(n+1)-14))^{\frac{1}{2}} \\ &\quad \times(\frac{1}{3}((-2-n)H_n^2+4(n+3)H_{n-1}^2-2^{n+1}(n+1)-17))^{\frac{1}{2}}.\end{aligned}$$

Proof.

(a) **(i)** We use equation (2.2).

(ii) We get

$$\begin{aligned}r_1(C) &= \max_i(\sum_j |c_{ij}|^2)^{\frac{1}{2}} = (\sum_{j=1}^{n} |c_{nj}|^2)^{\frac{1}{2}} \\ &= (\sum_{k=0}^{n-2} M_k^2+1)^{\frac{1}{2}} = (1+\sum_{k=0}^{n} M_k^2 - M_n^2 - M_{n-1}^2)^{\frac{1}{2}} \\ &= (\frac{1}{3}((-2-n)M_n^2+(4n+9)M_{n-1}^2-25M_0^2+18M_0M_1 \\ &\quad -3M_1^2+2(M_1-2M_0)(M_1-M_0)(2^n(n+1)-1))+1)^{\frac{1}{2}} \\ &= (\frac{1}{3}((-2-n)M_n^2+(4n+9)M_{n-1}^2+2^{n+1}(n+1)-2))^{\frac{1}{2}}\end{aligned}$$

and

$$\begin{aligned}c_1(D) &= \max_j(\sum_i |d_{ij}|^2)^{\frac{1}{2}} = (\sum_{i=1}^{n} |d_{i1}|^2)^{\frac{1}{2}} \\ &= (\sum_{k=0}^{n-1} W_k^2)^{\frac{1}{2}} = (\sum_{k=0}^{n} W_k^2 - W_n^2)^{\frac{1}{2}} \\ &= (\frac{1}{3}((-2-n)M_n^2+4(n+3)M_{n-1}^2-25M_0^2+18M_0M_1 \\ &\quad -3M_1^2+2(M_1-2M_0)(M_1-M_0)(2^n(n+1)-1)))^{\frac{1}{2}} \\ &= (\frac{1}{3}((-2-n)M_n^2+4(n+3)M_{n-1}^2+2^{n+1}(n+1)-5))^{\frac{1}{2}}\end{aligned}$$

so, from inequality (2.7),

$$\begin{aligned}\left\|A\right\|_2 &\leq r_1(C)c_1(D) = \Lambda_4 \\ &= (\frac{1}{3}((-2-n)M_n^2+(4n+9)M_{n-1}^2 \\ &\quad +2^{n+1}(n+1)-2))^{\frac{1}{2}} \times (\frac{1}{3}((-2-n)M_n^2+4(n+3)M_{n-1}^2 \\ &\quad +2^{n+1}(n+1)-5))^{\frac{1}{2}}.\end{aligned}$$

(b) **(i)** We use equation (2.2).

(ii) By definition, we get

$$\begin{aligned} r_1(C) &= \max_i (\sum_j |c_{ij}|^2)^{\frac{1}{2}} = (\sum_{j=1}^{n} |c_{nj}|^2)^{\frac{1}{2}} \\ &= (\sum_{k=0}^{n-2} H_k^2 + 1)^{\frac{1}{2}} = (1 + \sum_{k=0}^{n} H_k^2 - H_n^2 - H_{n-1}^2)^{\frac{1}{2}} \\ &= (\frac{1}{3}((-2-n)H_n^2 + (4n+9)H_{n-1}^2 - 25H_0^2 + 18H_0H_1 - 3H_1^2 \\ &\quad +2(H_1 - 2H_0)(H_1 - H_0)(2^n(n+1) - 1)) + 1)^{\frac{1}{2}} \\ &= (\frac{1}{3}((-2-n)H_n^2 + (4n+9)H_{n-1}^2 - 2^{n+1}(n+1) - 14))^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} c_1(D) &= \max_j (\sum_i |d_{ij}|^2)^{\frac{1}{2}} = (\sum_{i=1}^{n} |d_{i1}|^2)^{\frac{1}{2}} = (\sum_{k=0}^{n-1} H_k^2)^{\frac{1}{2}} = (\sum_{k=0}^{n} H_k^2 - H_n^2)^{\frac{1}{2}} \\ &= (\frac{1}{3}((-2-n)H_n^2 + 4(n+3)H_{n-1}^2 - 25H_0^2 + 18H_0H_1 - 3H_1^2 \\ &\quad +2(H_1 - 2H_0)(H_1 - H_0)(2^n(n+1) - 1)))^{\frac{1}{2}} \\ &= (\frac{1}{3}((-2-n)H_n^2 + 4(n+3)H_{n-1}^2 - 2^{n+1}(n+1) - 17))^{\frac{1}{2}} \end{aligned}$$

so, from inequality (2.7)

$$\begin{aligned} \|A\|_2 &\leq r_1(C)c_1(D) = \Lambda_5 \\ &= (\frac{1}{3}((-2-n)H_n^2 + (4n+9)H_{n-1}^2 - 2^{n+1}(n+1) - 14))^{\frac{1}{2}} \\ &\quad \times(\frac{1}{3}((-2-n)H_n^2 + 4(n+3)H_{n-1}^2 - 2^{n+1}(n+1) - 17))^{\frac{1}{2}}. \end{aligned}$$

This complates the proof. □

From the equation (2.10) and Corollary 7, we have the following corollary which gives the Frobenius norms of the Kronecker products of the Toeplitz matrices with special cases of generalized Mersenne numbers.

Corollary 9. *Suppose that $A = T(M_0, M_1, \cdots, M_{n-1})$ and $B = T(H_0, H_1, \cdots, H_{n-1})$ be Toeplitz matrices with Mersenne numbers and Mersenne-Lucas numbers respectively, then we have the following property:.*

$$\begin{aligned} \|A \otimes B\|_F &= \|A\|_F \|B\|_F \\ &= \sqrt{\Lambda_2}\sqrt{\Lambda_3} \end{aligned}$$

where Λ_2 and Λ_3 are as in Corollary 7 (a) and (b),

(set $W_n = M_n$ with $M_0 = 0, M_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 3$ respectively).

Proof. It can be easily seen from equation (2.10) and Theorem 6 and Corollary 7. □

From the above inequality (2.9) and Theorem 6 and Corollary 7, we have the following result, which gives an upper bound for the Frobenius norm of Hadamard products of Toeplitz matrices with different Mersenne sequences.

Corollary 10. *Assume that $A = T(M_0, M_1, \cdots, M_{n-1})$ and $B = T(H_0, H_1, \cdots, H_{n-1})$ be Toeplitz matrices with Mersenne numbers and Mersenne-Lucas numbers, respectively, then we have the following property:*

$$\begin{aligned} \|A \circ B\|_F &\leq \|A\|_F \|B\|_F \\ &\leq \sqrt{\Lambda_2}\sqrt{\Lambda_3} \end{aligned}$$

where Λ_2 and Λ_3 are as in Corollary 7 (a) and (b), (set $W_n = M_n$ with $M_0 = 0, M_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 3,$ respectively).

Proof. For the proof see inequality (2.9) and Theorem 6. □

In the last inequality (2.8) and Theorem 8, we have the following corollary, which gives an upper bound for the spectral norm of Hadamard products of Toeplitz matrices with different Mersenne sequences.

Corollary 11. *Given $A = T(M_0, M_1, \cdots, M_{n-1})$ and $B = T(H_0, H_1, \cdots, H_{n-1})$ be Toeplitz matrices with Mersenne numbers and Mersenne-Lucas numbers respectively, then we have the following property:*

$$\|A \circ B\|_2 \leq \Lambda_4 \times \Lambda_5$$

where Λ_4 and Λ_5 are as in Theorem 8,

(take $W_n = M_n$ with $M_0 = 0, M_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 3,$ respectively).

Proof. See inequality (2.8) and Theorem 8. □

From the related equation (2.10) and Theorem 8, we have the following corollary which gives an upper bound for the spectral norm of Kronocker products of Toeplitz matrices with different Mersenne sequences.

Corollary 12. *Let $A = T(M_0, M_1, \cdots, M_{n-1})$ and $B = T(H_0, H_1, \cdots, H_{n-1})$ be Toeplitz matrices with Mersenne numbers and Mersenne-Lucas numbers, respectively, then we have the following property:*

$$\|A \otimes B\|_2 \leq \Lambda_4 \times \Lambda_5$$

where Λ_4 and Λ_5 are as in Theorem 8,

(set $W_n = M_n$ with $M_0 = 0, M_1 = 1$ and $W_n = H_n$ with $H_0 = 2, H_1 = 3,$ respectively).

Proof. See equation (2.10) and Theorem 8. □

4 CONCLUSIONS

Recently, there have been so many studies of the sequences of numbers in the literature that concern about subsequences of the Horadam numbers which have second order recurrence relations. Generalized Mersenne numbers are special cases of Horadam numbers.

In this paper, we obtain results on Toeplitz matrices with Mersenne numbers components.

- In chapter 1, we present some known results on Mersenne numbers such as recurrence relation, characteristic equation and Binet's formulas.
- In chapter 2, we give some basic definitions and result of special norms of the Toeplitz matrices and find sum formulas of Toeplitz matrices with Mersenne numbers.
- In chapter 3, we obtain special norms of Toeplitz matrices with Mersenne numbers and find upper and lower bounds for spectral norms of Toeplitz matrices with Mersenne numbers components.

Linear recurrence relations (sequences) have many applications. Now, we present some applications of second order sequences.

- For the applications of Gaussian Fibonacci and Gaussian Lucas numbers to Pauli Fibonacci and Pauli Lucas quaternions, see [10].
- For the application of Pell Numbers to the solutions of three-dimensional difference equation systems, see [11].
- For the application of Jacobsthal numbers to special matrices, see [12].
- For the application of generalized k-order Fibonacci numbers to hybrid quaternions, see [13].
- For the applications of Fibonacci and Lucas numbers to Split Complex Bi-Periodic numbers, see [14].
- For the applications of generalized bivariate Fibonacci and Lucas polynomials to matrix polynomials, see [15].
- For the applications of generalized Fibonacci numbers to binomial sums, see [16].
- For the application of generalized Jacobsthal numbers to hyperbolic numbers, see [17].
- For the application of generalized Fibonacci numbers to dual hyperbolic numbers, see [18].
- For the application of Laplace transform and various matrix operations to the characteristic polynomial of the Fibonacci numbers, see [19].
- For the application of Generalized Fibonacci Matrices to Cryptography, see [20].
- For the application of higher order Jacobsthal numbers to quaternions, see [21].
- For the application of Fibonacci and Lucas Identities to Toeplitz-Hessenberg matrices, see [22].

- For the applications of Fibonacci numbers to lacunary statistical convergence, see [23].
- For the applications of Fibonacci numbers to lacunary statistical convergence in intuitionistic fuzzy normed linear spaces, see [24].
- For the applications of Fibonacci numbers to ideal convergence on intuitionistic fuzzy normed linear spaces, see [25].
- For the applications of k-Fibonacci and $k-$Lucas numbers to spinors, see [26].
- For the application of dual-generalized complex Fibonacci and Lucas numbers to Quaternions, see [27].
- For the application of special cases of Horadam numbers to Neutrosophic analysis see [28].
- For the application of Hyperbolic Fibonacci numbers to Quaternions, see [29].

COMPETING INTERESTS

Authors have declared that no competing interests exist.

REFERENCES

[1] Solak S, Bahşi M. On the spectral norms of toeplitz matrices with fibonacci and lucas numbers. Hacettepe Journal of Mathematics and Statistics. 2013;42(1);15-19.

[2] Akbulak M, Bozkurt D. On the norms of toeplitz matrices involving fibonacci and lucas numbers. Hacettepe Journal of Mathematics and Statistics. 2008;37(2):89-95.

[3] Shen S. On the norms of toeplitz matrices involving $k-$fibonacci and $k-$lucas numbers. Int. J. Contemp. Math. Sciences. 2012;7(8):363-368.

[4] Karpuz GE, Ateş F, Güngör AD, Cangül IN, Çevik AS. On the norms of toeplitz and hankel matrices with pell numbers. CP1281, ICNAAM, Numerical Analysis and Applied Mathematics, International Conference; 2010.

[5] Daşdemir A. On the norms of toeplitz matrices with the pell, pell-lucas and modfied pell numbers. Journal of Engineering Technology and Applied Sciences. 2016;1(2):51-57.

[6] Uygun Ş. On the bounds for the norms of toeplitz matrices with the jacobsthal and jacobsthal-lucas numbers. Journal of Engineering Technology and Applied Sciences. 2019;4(3):105-114.

[7] Uygun Ş, Aytar H. Bounds for the norms of toeplitz matrices with $k-$jacobsthal and $k-$jacobsthal-lucas numbers. Journal of Scientific Reports-A. 2020;45:90-100.

[8] Soykan Y. A study on generalized mersenne numbers. Journal of Progressive Research in Mathematics. 2021;18(3):90-108.

[9] Soykan Y. On the sum of the squares of generalized mersenne numbers: The sum formula $\sum_{k=0}^{n} x^k W_{mk+j}^2$. International Journal of Advances in Applied Mathematics and Mechanics. 2021;9(2);28-37.

[10] Azak AZ. Pauli gaussian fibonacci and pauli gaussian lucas quaternions. Mathematics. 2022;10:4655.
DOI: https://doi.org/10.3390/math10244655

[11] Büyük H, Taşkara N. On the solutions of three-dimensional difference equation systems via pell numbers. European Journal of Science and Technology. 2022;(Special Issue 34):433-440.

[12] Vasanthi S, Sivakumar B. Jacobsthal matrices and their properties. Indian Journal of Science and Technology. 2022;15(5):207-215.
DOI: https://doi.org/10.17485/IJST/v15i5.1948

[13] Gül K. Generalized k-order fibonacci hybrid quaternions. Erzincan University Journal of Science and Technology. 2022;15(2):670-683.
DOI: 10.18185/erzifbed.1132164

[14] Yılmaz N. Split complex Bi-Periodic fibonacci and lucas numbers. Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. 2022;71(1):153-164.
DOI:10.31801/cfsuasmas.704435

[15] Yılmaz N. The generalized bivariate fibonacci and lucas matrix polynomials. Mathematica Montisnigri. 2022;LIII:33-44.
DOI: 10.20948/mathmontis-2022-53-5

[16] Ulutaş YT, Toy D. Some equalities and binomial sums about the generalized fibonacci number u_n. Notes on Number Theory and Discrete Mathematics. 2022;28(2):252-260.
DOI: 10.7546/nntdm.2022.28.2.252-260

[17] Soykan Y, Taşdemir E. A study on hyperbolic numbers with generalized jacobsthal numbers Components. International Journal of Nonlinear

Analysis and Applications. 2022;13(2):1965-1981. DOI: http://dx.doi.org/10.22075/ijnaa.2021.22113.2328

[18] Soykan Y. On dual hyperbolic generalized fibonacci numbers. Indian J Pure Appl Math; 2021.
DOI: https://doi.org/10.1007/s13226-021-00128-2

[19] Deveci Ö, Shannon AG. On recurrence results from matrix transforms. Notes on Number Theory and Discrete Mathematics. 2022;28(4):589-592.
DOI: 10.7546/nntdm.2022.28.4.589-592

[20] Prasad K, Mahato H. Cryptography using generalized fibonacci matrices with affine-hill cipher. Journal of Discrete Mathematical Sciences & Cryptography. 2022;25(8-A):2341-2352.
DOI : 10.1080/09720529.2020.1838744

[21] Özkan E, Uysal M. On quaternions with higher order jacobsthal numbers components. Gazi University Journal of Science. 2023;36(1):336-347.
DOI: 10.35378/gujs. 1002454

[22] Goy T, Shattuck M. Fibonacci and lucas identities from toeplitz-hessenberg matrices. Appl. Appl. Math. 2019;14(2):699-715.

[23] Bilgin NG. Fibonacci lacunary statistical convergence of order γ in IFNLS. International Journal of Advances in Applied Mathematics and Mechanics. 2021;8(4):28-36.

[24] Kişi Ö, Tuzcuoglu I. Fibonacci lacunary statistical convergence in intuitionistic fuzzy normed linear spaces. Journal of Progressive Research in Mathematics. 2020;16(3);3001-3007.

[25] Kişi Ö, Debnathb P. Fibonacci ideal convergence on intuitionistic fuzzy normed linear spaces. Fuzzy Information and Engineering. 2022;1-13.
DOI: https://doi.org/10.1080/16168658.2022.2160226

[26] Kumari M, Prasad K, Frontczak R. On the k-fibonacci and $k-$lucas spinors. Notes on Number Theory and Discrete Mathematics. 2023;29(2):322-335.
DOI: 10.7546/nntdm.2023.29.2.322-335

[27] Şentürk GY, Gürses N, Yüce S. Construction of dual-generalized complex fibonacci and lucas quaternions. Carpathian Math. Publ. 2022;14(2):406-418.
DOI:10.15330/cmp.14.2.406-418

[28] Gökbaş H, Topal S, Smarandache F. Neutrosophic number sequences: An introductory study. International Journal of Neutrosophic Science (IJNS). 2023;20(01):27-48.
DOI: https://doi.org/10.54216/IJNS.200103

[29] Daşdemir A. On recursive hyperbolic fibonacci quaternions. Communications in Advanced Mathematical Sciences. 2021;4(4):198-207.
DOI:10.33434/cams.997824

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