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Hamiltonian Laceability in Ring Product and Cyclo Product of Graphs

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Abstract

B. Alspach, C.C. Chen and Kevin Mc Avaney [1] have discussed the Hamiltonian laceability of the Brick product $C(2n, m, r)$ for even cycles. In [2], the authors have shown that the (m, r)-Brick Product $C(2n + 1, 1, 2)$ is Hamiltonian-t-laceable for $1 \leq t \leq$ diam$n$. In [3] the authors have defined and discussed Hamiltonian-t-laceability properties of cyclic product $C(2n, m)$ cyclic product of graphs. In this paper we explore Hamiltonian-t^*-laceability of $(W_{1,n}, k)$ graph and Cyclo Product $C_y(n, mk)$ of graph.

Keywords: Brick product, Hamiltonian-t-laceable graph, Cyclo product.
2000 Mathematics Subject Classification: 05CS45; 05CS99.

1 Introduction

Let G be a finite, simple, connected and undirected graph. Let u and v be two vertices in G. The distance between u and v denoted by, $d(u, v)$, is the length of a shortest $u - v$ path in G. A graph G is Hamiltonian-t-laceable [4] if there exists in G Hamiltonian path between every pair of vertices u and v with $d(u, v)$=t, $1 \leq t \leq diamG$, where t is a positive integer. A graph G is Hamiltonian-t^*-laceable [2] if there exists in G a Hamiltonian path between at least one pair of distinct vertices u and v such that $d(u, v)$=t, $1 \leq t \leq diamG$. In [1] B. Alspach, C.C. Chen and Kevin McAvaney have explored Hamiltonian Laceability in the Brick Products of even cycles. In [3], Leena Shenoy and R. Murali have discussed the (m, r)-Brick Product of odd cycles $C(2n + 1, m, r)$. Using this concept we define $(W_{1,n}, k)$ graph and Cyclo product, $C_y(n, mk)$ of graph and explore Hamiltonian laceability properties.
First we recall the following definitions.

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2 Hamiltonian Laceable Graph

Definition 2.1. Let G be a finite, simple, connected undirected graph. A graph G is $Hamiltonian$ $laceable$ if there exists a Hamiltonian path between every pair of vertices at an odd distance in G.

Example: Hamiltonian laceable graph is shown in figure 1.

Definition 2.2. Let G be a finite, simple, connected undirected graph. The graph G is $Hamiltonian-t-laceable$ if there exist a Hamiltonian path between every pair of distinct vertices a_i and a_j in G with the property $d(a_i, a_j) = t$; $1 \leq t \leq diamG$ and $Hamiltonian - t^* - laceable$ if there exists a Hamiltonian path between at least any one pair of distinct vertices a_i and a_j such that $d(a_i, a_j)$ = t; $1 \leq t \leq diamG$.

Example: $Hamiltonian - 3 - laceable$ and $Hamiltonian - 2^* - laceable$ graphs are shown in figure 2.

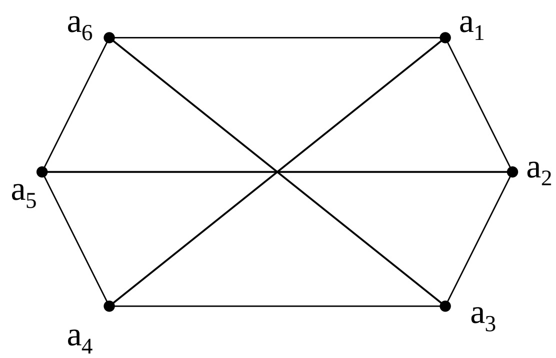

Figure 1: A Hamiltonian laceable graph

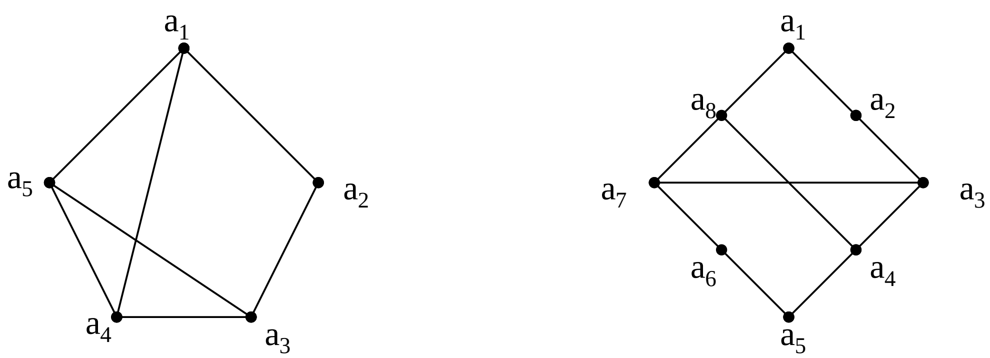

Figure 2: Hamiltonian-2^*- laceable graph and Hamiltonian-3-laceable graph

3 The Graph $(W_{1,n}, k)$

Let $W_{1,n}$ be a wheel graph. We shall denote the vertices of $W_{1,n}$ by $< a_i >$, $1 \le i \le n$ and a root vertex a_0.
Let $G = (W_{1,n}, k)$ be a graph obtained by taking disjoint union of k copies of cycle C_k with the vertices $a_{k_1}, a_{k_2}, a_{k_3}, a_{k_3}........a_{k_n}$.
If k=1 , For $1 \le i \le n$, draw an edge connecting vertices a_i of $W_{1,n}$ to a_{1_i} of C_1.
If $k \ge$2, starting from k=2 proceed recursively joining the vertices $a_{(k-1)_i}$ to a_{k_i} by an edge. Where $1 \le i \le n$.
Example: The graph $(W_{1,n}, k)$ is shown in figure 3.

Theorem 3.1. *The graph $G = (W_{1,n}, k)$, $n \ge 3, k \ge 1$ is Hamiltonian-t^*-laceable for $1 \le t \le 3$.*

Proof. Let $G = (W_{1,n}, k)$. The vertices of interior cycle $(W_{1,n})$ be $a_1, a_2, a_3, a_4.........a_{n-1}, a_n$ and vertices on the k^{th} cycle be $a_{k_1}, a_{k_2}, a_{k_3}, a_{k_4}.............a_{k_{n-1}}, a_{k_n}$. The graph G has $n(k + 1) + 1$ number of vertices and $2(k + 1)n$ number of edges. To establish the result, we consider the following cases,
Case(i): For t=1
In G, let $d(a_0, a_1)$= 1 then the path
$P : (a_0, a_2) \cup \{(a_2, a_3) \cup (a_3, a_4) \cup .... \cup (a_{n-1}, a_n)\} \cup (a_n, a_{1_n}) \cup \{(a_{1_n}, a_{1_{(n-1)}}) \cup (a_{1_{n-1}}, a_{1_{(n-2)}}) \cup .... \cup (a_{1_3}, a_{1_2})\} \cup (a_{1_2}, a_{2_2}) \cup \{(a_{2_2}, a_{2_3}) \cup (a_{2_3}, a_{2_4}) \cup (a_{2_4}, a_{2_5}) \cup .... \cup (a_{2_{(n-1)}}, a_{2_n})\} \cup (a_{2_n}, a_{3_n}) \cup \{(a_{3_n}, a_{3_{n-1}}) \cup (a_{3_{n-1}}, a_{3_{n-2}}) \cup (a_{3_{n-2}}, a_{3_{n-3}}) \cup .... \cup (a_{3_3}, a_{3_2})\} \cup (a_{3_2}, a_{4_2}) \cup .... \cup T \cup (a_{k_1}, a_{(k-1)_1}) \cup (a_{(k-1)_1}, a_{(k-2)_1}) \cup .... \cup (a_{2_1}, a_{1_1}) \cup (a_{1_1}, a_1)$ is a Hamiltonian path from a_0 to a_1.
Where,

$$T = \begin{cases} (a_{k_n}, a_{k_{n-1}}) \cup (a_{k_{n-1}}, a_{k_{n-2}}) \cup .... \cup (a_{k_3}, a_{k_2}) \cup (a_{k_2}, a_{k_1}), & \text{if } k \text{ is odd ;} \\ (a_{k_2}, a_{k_3}) \cup (a_{k_3}, a_{k_4}) \cup .... \cup (a_{k_{n-1}}, a_{k_n}) \cup (a_{k_n}, a_{k_1}), & \text{if } k \text{ is even .} \end{cases}$$

Case(ii): For t=2
In G, let $d(a_0, a_{1_1})$= 2 then the path
$P : (a_0, a_1) \cup \{(a_1, a_2) \cup (a_2, a_3) \cup (a_3, a_4) \cup .... \cup (a_{n-2}, a_{n-1}) \cup (a_{n-1}, a_n)\} \cup (a_n, a_{1_n}) \cup \{(a_{1_n}, a_{1_{(n-1)}}) \cup (a_{1_{(n-1)}}, a_{1_{(n-2)}}) \cup (a_{1_{n-2}}, a_{1_{(n-3)}}) \cup .... \cup (a_{1_4}, a_{1_3}) \cup (a_{1_3}, a_{1_2})\} \cup (a_{1_2}, a_{2_2}) \cup \{(a_{2_2}, a_{2_3}) \cup (a_{2_3}, a_{2_4}) \cup (a_{2_4}, a_{2_5}) \cup .... \cup (a_{2_{(n-1)}}, a_{2_n})\} \cup (a_{2_n}, a_{3_n}) \cup \{(a_{3_n}, a_{3_{n-1}}) \cup (a_{3_{n-1}}, a_{3_{n-2}}) \cup (a_{3_{n-2}}, a_{3_{n-3}}) \cup .... \cup (a_{3_4}, a_{3_3}) \cup (a_{3_3}, a_{3_2}) \cup (a_{3_2}, a_{4_2}) \cup .... \cup T \cup (a_{k_1}, a_{(k-1)_1}) \cup (a_{(k-1)_1}, a_{(k-2)_1}) \cup .... \cup (a_{2_1}, a_{1_1})$ is a Hamiltonian path from a_0 to a_{1_1}. Where

$$T = \begin{cases} (a_{k_n}, a_{k_{(n-1)}}) \cup (a_{k_{(n-1)}}, a_{k_{(n-2)}}) \cup .... \cup (a_{k_3}, a_{k_2}) \cup (a_{k_2}, a_{k_1}), & \text{if } k \text{ is odd ;} \\ (a_{k_2}, a_{k_3}) \cup (a_{k_3}, a_{k_4}) \cup .... \cup (a_{k_{(n-1)}}, a_{k_n}) \cup (a_{k_n}, a_{k_1}), & \text{if } k \text{ is even .} \end{cases}$$

Case(iii): For t=3
In G, let $d(a_3, a_{1_1})$= 3 then the path
$P : \{(a_3, a_4) \cup \{(a_4, a_5) \cup (a_5, a_6) \cup .... \cup (a_n, a_1)\} \cup (a_1, a_0)\} \cup (a_0, a_2) \cup \{(a_2, a_{1_2}) \cup \{(a_{1_2}, a_{1_3}) \cup (a_{1_3}, a_{1_4}) \cup (a_{1_4}, a_{1_5}) \cup .... \cup (a_{1_{(n-1)}}, a_{1_n})\} \cup (a_{1_n}, a_{2_n}) \cup \{(a_{2_n}, a_{2_{n-1}}) \cup \{(a_{2_{n-1}}, a_{2_{n-2}}) \cup .... \cup (a_{2_4}, a_{2_3}) \cup (a_{2_3}, a_{2_2})\} \cup (a_{2_2}, a_{3_2}) \cup \{(a_{3_2}, a_{3_3}) \cup \{(a_{3_3}, a_{3_4}) \cup (a_{3_4}, a_{3_5}) \cup .... \cup (a_{3_{n-1}}, a_{3_n})\} \cup (a_{3_n}, a_{4_n}) \cup .... \cup T \cup (a_{k_1}, a_{(k-1)_1}) \cup (a_{(k-1)_1}, a_{(k-2)_1}) \cup .... \cup (a_{2_1}, a_{1_1})$ is a Hamiltonian path from a_3 to a_{1_1}. Where

$$T = \begin{cases} (a_{k_2}, a_{k_3}) \cup (a_{k_3}, a_{k_4}) \cup .... \cup (a_{k_{(n-1)}}, a_{k_n}) \cup (a_{k_n}, a_{k_1}), & \text{if } k \text{ is odd ;} \\ (a_{k_n}, a_{k_{(n-1)}}) \cup (a_{k_{(n-1)}}, a_{k_{(n-2)}}) \cup .... \cup (a_{k_3}, a_{k_2}) \cup (a_{k_2}, a_{k_1}), & \text{if } k \text{ is even .} \end{cases}$$

Hence the proof. □

Figure 4, shows a Hamiltonian path between a_0 to a_{1_1} in $(W_{1,8}, 3)$. This path is
$P : (a_0, a_1) \cup (a_1, a_2) \cup (a_2, a_3) \cup .... \cup (a_7, a_8) \cup (a_8, a_{1_8}) \cup (a_{1_8}, a_{1_7}) \cup (a_{1_7}, a_{1_6}) \cup .... \cup (a_{1_3}, a_{1_2}) \cup (a_{1_2}, a_{2_2}) \cup (a_{2_2}, a_{2_3}) \cup (a_{2_3}, a_{2_4}) \cup .... \cup (a_{2_7}, a_{2_8}) \cup (a_{2_8}, a_{3_8}) \cup (a_{3_8}, a_{3_7}) \cup (a_{3_7}, a_{3_6}) \cup (a_{3_1}, a_{2_1}) \cup (a_{2_1}, a_{1_1})$.

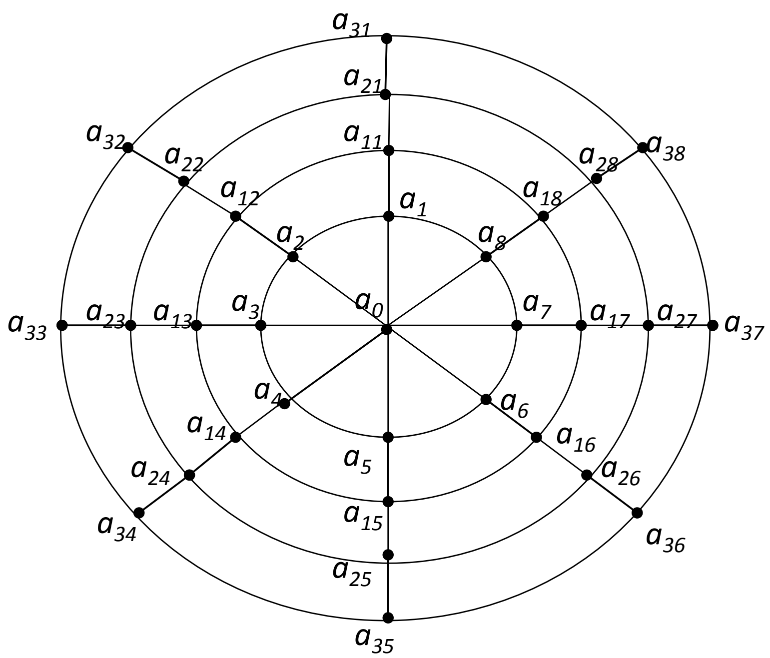

Figure 3: Hamiltonian laceable graph $(W_{1,8}, 3)$

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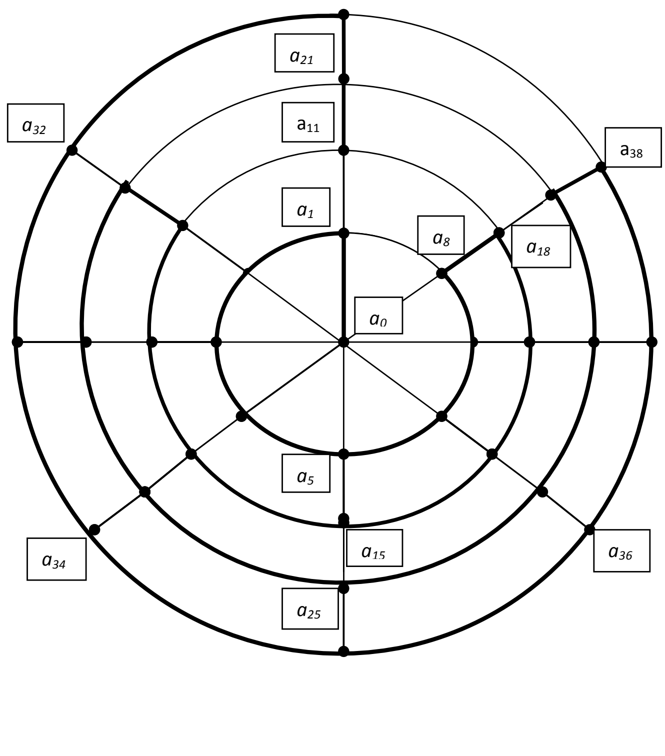

Figure 4: Hamiltonian path from a_0 to a_{1_1} is shown by dark lines in $(W_{1,8}, 3)$

4 Cyclo Product

Definition 4.1. Let n, k be positive integers and $m \geq 2$. The $cycloproduct$ $C_y(n, mk)$ is defined by joining each vertex a_i in C_n to a_{i+mk} under modulo n.

Cyclo product $C_y(15, 3k)$ under modulo 15 is shown in fig 5.

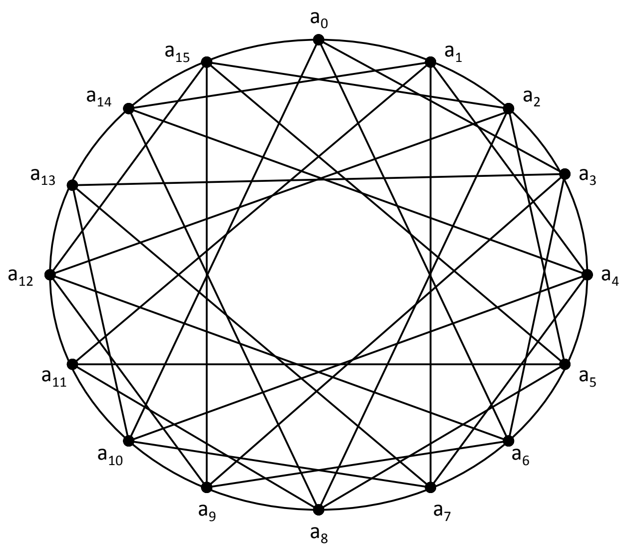

Figure 5: Cyclo product $C_y(15, 3k)$ under modulo 15

Theorem 4.1. *The graph* $C_y(n, 3k)$, $n \geq 6$ *is Hamiltonian-*t*-laceable for* t *=1, 2.*

Proof. Let G= $C_y(n, 3k)$ be a graph with n no. vertices. Let this vertex set be V={$a_0, a_1, a_2, a_3, a_4, a_5$.. ......$a_{n-2}, a_n$}. We consider the following cases
Case(i): For t=1.
Let $d(a_i, a_j)$=1 and | $i - j$ | =1 then we find a path P in G such that P : $(a_i, a_{i-1}) \cup (a_{i-1}, a_{i-2}) \cup (a_{i-2}, a_{i-3}) \cup (a_{i-3}, a_{i-4}) \cup$ .... $\cup (a_{j+2}, a_{j+1}) \cup (a_{j+1}, a_j)$ is a Hamiltonian path.
Hence G is Hamiltonian-1-laceable.
Case(ii): For t=2.
Let $d(a_i, a_j)$=2 and | $i - j$ | =2 then we find a path P in G such that P: $(a_i, a_{i+1}) \cup (a_{i+1}, a_{i-5}) \cup (a_{i-5}, a_{i-4}) \cup (a_{i-4}, a_{i-1}) \cup (a_{i-1}, a_{i-2}) \cup (a_{i-2}, a_{i-3}) \cup (a_{i-3}, a_{i-6}) \cup (a_{i-6}, a_{i-7}) \cup (a_{i-7}, a_{i-8}) \cup (a_{i-8}, a_{i-9}) \cup (a_{i-9}, a_{i-10}) \cup$ .... $\cup (a_{j+2}, a_{j+1}) \cup (a_{j+1}, a_j)$ is a Hamiltonian path from a_i to a_j.
Hence G is Hamiltonian-2-laceable.
Hence the proof. □

In figure 6, a Hamiltonian path in G = $C_y(15, 3k)$ under modulo 15, between the vertices a_1 to a_3 is shown. This path is P : $(a_1, a_2) \cup (a_2, a_{12}) \cup (a_{12}, a_{13}) \cup (a_{13}, a_0) \cup (a_0, a_{15}) \cup (a_{15}, a_{14}) \cup (a_{14}, a_{11}) \cup (a_{11}, a_{10}) \cup (a_{10}, a_9) \cup (a_9, a_8) \cup (a_8, a_7) \cup (a_7, a_6) \cup (a_6, a_5) \cup (a_5, a_4) \cup (a_4, a_3)$.

Theorem 4.2. *The graph* G= $C_y(n, 2k)$, $n \geq 6$ *is Hamiltonian-*t*-laceable for* t*=1,2.*

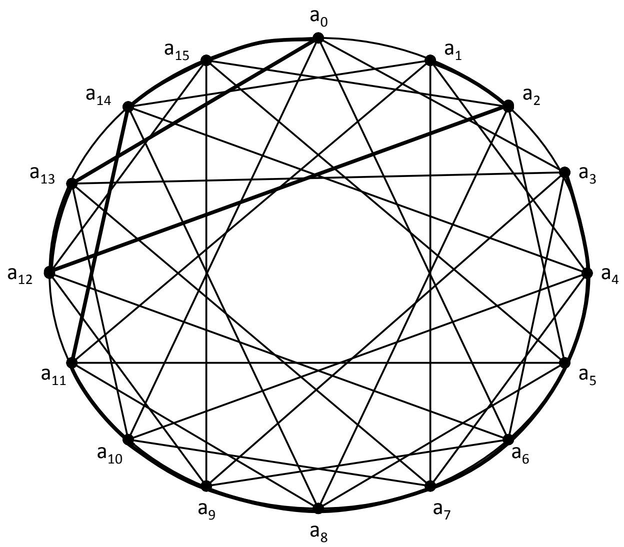

Figure 6: Hamiltonian path from a_1 to a_3 is shown by dark lines in $C_y(15, 3k)$

Proof. Let G=$C_y(n, 3k)$ be the graph of order n, $(n \geq 6)$. Let the vertex set $V = \{a_0, a_1, a_2, a_3, a_4, a_5...$ $.....a_{n-2}, a_n\}$

Case(i): For t=1

Let $d(a_i, a_j)$=1 and $\mid i - j \mid$=2,$0 \leq i, j \leq n$. There exists a path P in G such that $P : (a_i, a_{i+3}) \cup (a_{i+3}, a_{i+5}) \cup (a_{i+5}, a_{i+7}) \cup (a_{i+7}, a_{i+9}) \cup (a_{i+9}, a_{i+11}) \cup (a_{i+11}, a_{i+12}) \cup .... \cup (a_{i-1}, a_{j+1}) \cup (a_{j+1}, a_{i-2}) \cup (a_{i-2}, a_{i-4}) \cup (a_{i-4}, a_{i-8}) \cup .... \cup (a_{i-1}, a_{j+1}) \cup (a_{j+1}, a_{i-2}) \cup (a_{i-2}, a_{i-4}) \cup (a_{i-4}, a_{i-8}) \cup .... \cup (a_{j+4}, a_{j+2}) \cup (a_{j+2}, a_j)$ is a Hamiltonian path from a_i to a_j under modulo n. Hence G is Hamiltonian-1-laceable.

Case(ii): For t=2.

Let $d(a_i, a_j)$=2 and $j - i$=1, $0 \leq i, j \leq n$. There exists a path P in G such that $P : (a_i, a_{i+1}) \cup (a_{i+1}, a_{i+2}) \cup (a_{i+2}, a_{i-2}) \cup (a_{i-2}, a_{i+1}) \cup (a_{i+1}, a_{i-3}) \cup (a_{i-3}, a_{i-4}) \cup (a_{i-4}, a_{i-5}) \cup (a_{i-5}, a_{i-6}) \cup (a_{i-6}, a_{i-7}) \cup .... \cup (a_{j+3}, a_{j+2}) \cup (a_{j+2}, a_{j+1}) \cup (a_{j+1}, a_j)$ is a Hamiltonian path
Hence Hamiltonian-2-laceable. □

In figure 7, a Hamiltonian path in G = $C_y(15, 2k)$ under modulo 15, between the vertices a_1 to a_4 is shown. This path is $P : (a_1, a_2) \cup (a_2, a_3) \cup (a_3, a_{14}) \cup (a_{14}, a_0) \cup (a_0, a_{13}) \cup (a_{13}, a_{12}) \cup (a_{12}, a_{11}) \cup (a_{11}, a_{10}) \cup (a_{10}, a_9) \cup (a_9, a_8) \cup (a_8, a_7) \cup (a_7, a_6) \cup (a_6, a_5) \cup (a_5, a_4)$.

5 Laceability in Square of a Graph

Let G be a simple connected graph with n vertices. G^2 of G is the graph obtained by inserting edge between every two vertices u and v at a distace $d(u, v)$=2.

Theorem 5.1. *If* $G = C(2n, 1)$ *then* $G^2 - G$ *is Hamiltonian-*t^**-laceable for* t*=1,2.*

Proof. Let $G = C(2n, 1)$ be a graph of order n. Let $G^2 - G$ is the graph having same vertex as in G. Consider the following cases

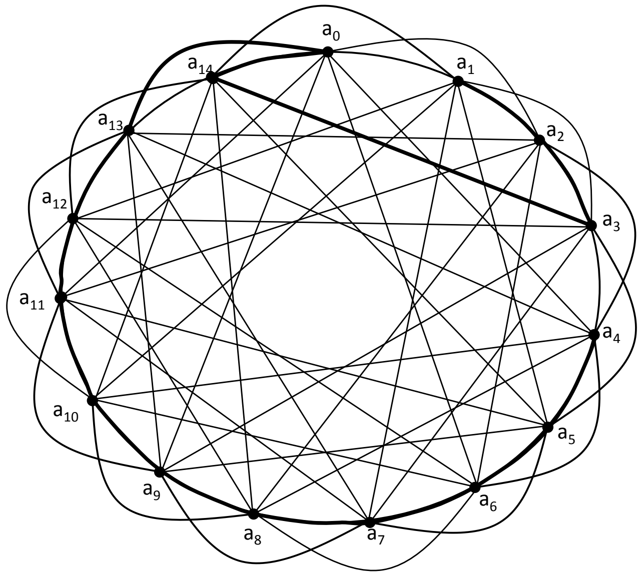

Figure 7: Hamiltonian path from a_1 to a_4 is shown by dark lines in $C_y(15, 3k)$

Case(i): For t=1
Let $d(a_i, a_j)$=1 where i=0 and $j = \frac{n+2}{2}$ then there exists a path$P : a_i J[P]^{\frac{n-4}{2}} JJ[P]^{\frac{n-4}{2}}$ is a Hamiltonian path from $a_i to a_j$. Hence G_1 is Hamiltonian-1^*-laceable.
Case(ii):For t=2
Let $d(a_i, a_j)$=2 where i=0 and $j = \frac{n-4}{2}$
$P : a_i JP^{\frac{n-4}{2}} JJ[P]^{\frac{n}{2}} KJ$ is a Hamiltonian path from a_i to a_j. Hence G_1 is Hamiltonian-2^*-laceable.
□

6 Conclusions

In this paper we have explored Hamiltonian properties of Cyclo product, the graph $G = (W_{1,n}, k)$ and square of a graph.

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Competing Interests

Hamiltonian-t^*-laceability properties of Cyclo product and the graph $G = (W_{1,n}, k)$ can be obtained for higher values of t.

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