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Revolution Surfaces with Constant Mean Curvature in Non-Euclidean Spaces

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Abstract

In this article, we study the conformal mean curvature equation in Thurston's geometries of Sol space. The classification of revolution surfaces with mean curvature was obtained by studying the corresponding profile curves in Sol space. According to the characteristics of the conformal metric, the revolution surfaces in Sol manifold were obtained through a profile curve revolving respectively. Assumes that the mean curvatures of these revolution surfaces were certain functions, the corresponding differential equations about the profile curves can be obtained. By solving these differential equations, the classification of the revolution surfaces with conformal mean curvature was achieved.

Keywords: Sol manifold; revolution surface; mean curvature; conformal metric.

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1 Introduction

With the development of mathematics, hyperbolic geometry has become an important branch of mathematics. Hopf conjecture [1] states that a compact surface immersed in R^n with constant mean curvature (CMC) is the standard (round) sphere. It can be viewed as a generalization of Alexandrov's theorem which asserts that every compact embedded CMC surface in R^3 is the round sphere. This conjecture has been disproved by Hsiang [2] who constructed a counterexample in R^4 and then by Wente [3] who produced an immersion of a compact oriented two-dimensional

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surface of genus 1 into R^3 with constant mean curvature. Finally Kapouleas ([4,5]) constructed examples of CMC surfaces for every genus $g \geq 2$. In non-Euclidean manifold, Thurston Sol manifold geometry is the study of a wide range of space. Because it has the same with Euclidean space, mathematics workers have done a lot of research work [6-8]. Kenmotsu respectively discussed constant mean curvature surfaces in R^3 and the given mean curvature revolution surfaces in R^3 [9]. We have not discussed revolution surfaces with given mean curvature function in Thurston's geometries of Sol space. In this note, we will prove the existence of revolution surfaces with conformal mean curvature in Sol space.

Theorem: For every rotationally invariant compact smooth surface S embedded in Thurston's geometries of Sol space there exists a conformally flat metric g of R^3 such that S has constant mean curvature with respect to g.

Sol space, one of Thurston's eight 3-dimensional geometries, can be viewed as R^3 provided with Riemannian metric $g_{Sol} = ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$, where (x,y,z) are the standard coordinates in R^3. Note the Sol metric can also be written as:

$$ds^2 = \sum_{i=1}^{3} \omega_i \otimes \omega_i,$$

where

$$\omega_1 = e^z dx, \omega_2 = e^{-z} dy, \omega_3 = dz,$$

and the orthonormal basis dual to the 1-form is

$$e_1 = e^{-z}\frac{\partial}{\partial x}, e_2 = e^z\frac{\partial}{\partial y}, e_3 = \frac{\partial}{\partial z},$$

With respect to this orthonormal basis, the Levi-Civita connection and the Lie brackets can be easily computed as:

$$\nabla_{e_1}^{e_1} = -e_3, \nabla_{e_1}^{e_2} = 0, \nabla_{e_1}^{e_3} = e_1,\ \nabla_{e_2}^{e_1} = 0, \nabla_{e_2}^{e_2} = e_3, \nabla_{e_2}^{e_3} = -e_2,\ \nabla_{e_3}^{e_1} = 0, \nabla_{e_3}^{e_2} = 0, \nabla_{e_3}^{e_3} = 0,$$

$$[e_1, e_2] = 0, [e_2, e_3] = -e_2, [e_1, e_3] = e_1.$$

We adopt the following notation and sign convention for Riemannian curvature operator.

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

The Riemannian curvature tensor is given by

$$R(X,Y,Z,W) = g(R(Y,X)Z,W) = -g(R(X,Y)Z,W),$$

A direct computation using the formula gives the following non-zero components of Riemannian curvature of Sol space with respect to the orthonormal basis $\{e_1, e_2, e_3\}$:

$$R_{121} = -e_2, R_{131} = e_3, R_{122} = e_1,\ R_{232} = e_3, R_{133} = -e_1, R_{233} = -e_2,$$

and

$$R_{1212} = -g(R(e_1,e_2)e_1,e_2) = -g(-e_2,e_2) = 1,$$

$$R_{1313} = -g(R(e_1,e_3)e_1,e_3) = -g(e_3,e_3) = -1,$$

$$R_{2323} = -g(R(e_2,e_3)e_2,e_3) = -g(e_3,e_3) = -1.$$

The proof is done by solving for g the equation $H_g = c$, where H_g is the mean curvature of S in (R^3, g) [10], which we will compute with the formula

$$H_g = \frac{1}{2}[g\left(\nabla_{e_1}^{e_1}, v\right) + g\left(\nabla_{e_2}^{e_2}, v\right)].$$

Hence we first need to find an orthonormal basis $\{e_1, e_2, v\}$ for (R^3, g) (such that $\{e_1, e_2\}$ is an orthonormal basis for the tanget space of S) and the covariant derivatives $\nabla_{e_1}^{e_1}$ and $\nabla_{e_2}^{e_2}$ for which we need the Christoffel symbols Γ_{ij}^{m}.

2 Preliminaries

In R^3 consider the cylindrical coordinates (x, ρ, θ) corresponding to the cartesian coordinates $(x, y = \rho\cos\theta, z = \rho\sin\theta)$ [5]. The surface of revolution S obtained by rotating the graph of the function $r(x)$ around the x-axis is given by the immersion:

$$(x,\theta) \overset{X}{\mapsto} \left(x, r(x), \theta\right),$$

i.e.

$$S = \left\{(x,\rho,\theta) \in [x_1, x_2] \times R^+ \times [0, 2\pi] \middle| \rho = r(x)\right\}$$

for some $x_1 < x_2$, where the following closing condition holds:

$$r(x) \geq 0, \forall x \in [x_1, x_2], r(x_1) = 0 = r(x_2).$$

Moreover since S is supposed to be smooth, the tanget line to $r(x)$ at $x_i, i = 1,2$ has to be vertical; finally to avoid self-intersections the two endoponits x_i are the only points where r vanishes. In this coordinate system, the euclidean metric has matrix.

$$\varepsilon = \left(\varepsilon_{ij}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \rho^2 \end{pmatrix},$$

in fact:

$$\partial_\rho = \cos\theta\partial_y + \sin\theta\partial_z, \partial_\theta = -\rho\sin\theta\partial_y + \rho\cos\theta\partial_z,$$

We will modify the euclidean metric of R^3 by adding to it a rotationally invariant smooth function $M = M(x, \rho) : R \times R^+ \to R$. The new metric will be

$$g_{ij} = \begin{pmatrix} e^{f(x,\rho)} & 0 & 0 \\ 0 & e^{f(x,\rho)} & 0 \\ 0 & 0 & \rho^2 e^{f(x,\rho)} \end{pmatrix} = e^{f(x,\rho)}\varepsilon,$$

Note that $g = e^{f(x,\rho)}\varepsilon$, hence the new metric is conformal to the euclidean one.

We will need a basis $\{e_1, e_2, v\}$ for R^3, orthonormal in the metric g, such that $\{e_1, e_2\}$ is an orthonormal basis for the tanget space T_pS at the point $p = X(x, v)$. As usual we will obtain $\{e_1, e_2\}$ by normalizing the two vectors $\{\tilde{e}_1, \tilde{e}_2\}$ that generate T_pS, which are

$$\tilde{e}_1 = \frac{\partial X}{\partial x} = (1, r', 0), \tilde{e}_2 = \frac{\partial X}{\partial \theta} = (0,0,1),$$

Hence

$$g(\tilde{e}_1, \tilde{e}_1) = e^{f(x,\rho)}(1 + r'^2), g(\tilde{e}_2, \tilde{e}_2) = e^{f(x,\rho)}\rho^2,$$

which yield

$$e_1 = \frac{1}{\sqrt{e^{f(x,\rho)}(1 + r'^2)}}(1, r', 0), e_2 = \frac{1}{\rho\sqrt{e^{f(x,\rho)}}}(0,0,1),$$

In the last expression we used $\{\partial_x, \partial_\rho, \partial_\theta\}$ as a basis for T_pS. To find v we can use the vector product as in the euclidean case, since a conformal change in the meric does respect the angles, hence:

$$v = \frac{e_1 \times e_2}{\|e_1 \times e_2\|} = \frac{1}{\sqrt{e^{f(x,\rho)}(1 + r'^2)}}(r', -1, 0).$$

3 Christoffel Symbols

We are now in the position to compute the Christoffel symbols for the connection induced on S by the metric g; we will adopt the notation

$$x = x^1, \rho = x^2, \theta = x^3,$$

and use the formula [6]

$$\Gamma_{ij}^{m}=\frac{1}{2}\sum_{k}g^{km}\left(\frac{\partial}{\partial x^{i}}g_{jk}+\frac{\partial}{\partial x^{j}}g_{ki}-\frac{\partial}{\partial x^{k}}g_{ij}\right),$$

where as usual (g^{ij}) is the inverse matrix of (g_{ij}), so in the cylindrical coordinates

$$g^{ij}=\left(g_{ij}\right)^{-1}=\begin{pmatrix} e^{-f(x,\rho)} & 0 & 0 \\ 0 & e^{-f(x,\rho)} & 0 \\ 0 & 0 & \frac{1}{\rho^{2}}e^{-f(x,\rho)} \end{pmatrix},$$

For $m=1$ the above formula simplifies to:

$$\Gamma_{ij}^{1}=\frac{1}{2}g^{11}\left(\partial_{i}g_{i1}+\partial_{j}g_{j1}-\partial_{x}g_{ij}\right),$$

So we get:

$$\Gamma_{11}^{1}=\frac{1}{2}f_{x},\Gamma_{12}^{1}=\frac{1}{2}f_{\rho}=\Gamma_{21}^{1},\Gamma_{13}^{1}=0=\Gamma_{31}^{1},\Gamma_{23}^{1}=0=\Gamma_{32}^{1},\Gamma_{22}^{1}=-\frac{1}{2}f_{x},\Gamma_{33}^{1}=-\frac{1}{2}\rho^{2}f_{x}.$$

In the same way for $m=2$ the formula becomes

$$\Gamma_{ij}^{2}=\frac{1}{2}g^{22}\left(\partial_{i}g_{j2}+\partial_{j}g_{2i}-\partial_{r}g_{ij}\right),$$

and we obtain:

$$\Gamma_{11}^{2}=-\frac{1}{2}f_{\rho},\Gamma_{12}^{2}=\frac{1}{2}f_{x}=\Gamma_{21}^{2},\Gamma_{23}^{2}=0=\Gamma_{32}^{2},\Gamma_{13}^{2}=0=\Gamma_{31}^{2},\Gamma_{22}^{2}=\frac{1}{2}f_{\rho},\Gamma_{33}^{2}=-\frac{1}{2}\left(2\rho+\rho^{2}f_{\rho}\right).$$

Finally, for $m=3$

$$\Gamma_{ij}^{3}=\frac{1}{2}g^{33}\left(\partial_{i}g_{j3}+\partial_{j}g_{3i}-\partial_{\theta}g_{ij}\right),$$

that gives:

$$\Gamma_{ij}^{3}=0,i,j\neq 3,\Gamma_{13}^{3}=\frac{1}{2}f_{x}=\Gamma_{31}^{3},\Gamma_{23}^{3}=\frac{2\rho+\rho^{2}f_{\rho}}{2\rho^{2}}=\Gamma_{32}^{3},\Gamma_{33}^{3}=0.$$

4 Covariant Derivatives and the Mean Curvature

To compute the mean curvature H_g of the surface S in the metric g we will use the formula [11]:

$$H_{g}=\frac{1}{2}[g\left(\nabla_{e_{1}}^{e_{1}},v\right)+g\left(\nabla_{e_{2}}^{e_{2}},v\right)].$$

where ∇ is the Levi-Civita connection of (R^3, g). To simplify the computations we are going to adopt the notation:

$$e_1 = \sum_{i=1}^{3} E_i \partial_i, e_2 = \sum_{i=1}^{3} F_i \partial_i,$$

$$\nabla_{e_1}^{e_1} = \sum_k \left(\sum_{ij} E^i E^j \Gamma_{ij}^k + e_1(E^k) \right) \partial_k = \left(E^1 E^1 \Gamma_{11}^1 + E^2 E^2 \Gamma_{22}^1 + E^3 E^3 \Gamma_{33}^1 + 2E^1 E^2 \Gamma_{12}^1 + e_1(E^1) \right) \partial_x$$
$$+ \left(2E^1 E^2 \Gamma_{12}^2 + E^1 E^1 \Gamma_{11}^2 + E^2 E^2 \Gamma_{22}^2 + E^3 E^3 \Gamma_{33}^2 + e_1(E^2) \right) \partial_\rho$$
$$+ \left(2E^1 E^3 \Gamma_{13}^3 + 2E^2 E^3 \Gamma_{23}^3 + e_1(E^3) \right) \partial_\theta,$$

$$e_1(E^1) = \left(E^1 \partial_1 + E^2 \partial_2 + E^3 \partial_3 \right)(E^1)$$
$$= \frac{1}{\sqrt{e^{f(x,\rho)}(1 + r'^2)}} \partial_x \left(\frac{1}{\sqrt{e^{f(x,\rho)}(1 + r'^2)}} \right) + \frac{r'}{\sqrt{e^{f(x,\rho)}(1 + r'^2)}} \partial_\rho \left(\frac{1}{\sqrt{e^{f(x,\rho)}(1 + r'^2)}} \right),$$

$$e_1(E^2) = \frac{1}{\sqrt{e^{f(x,\rho)}(1 + r'^2)}} \partial_x \left(\frac{r'}{\sqrt{e^{f(x,\rho)}(1 + r'^2)}} \right) + \frac{r'}{\sqrt{e^{f(x,\rho)}(1 + r'^2)}} \partial_\rho \left(\frac{r'}{\sqrt{e^{f(x,\rho)}(1 + r'^2)}} \right),$$

$$e_1(E^3) = 0,$$

where the partial derivatives are:

$$\partial_x \left(\frac{1}{\sqrt{e^{f(x,\rho)}(1 + r'^2)}} \right) = -\frac{f_x(1 + r'^2) + 2r'r''}{2\sqrt{e^{f(x,\rho)}(1 + r'^2)}^3}, \ \partial_r \left(\frac{1}{\sqrt{e^{f(x,\rho)}(1 + r'^2)}} \right) = -\frac{f_\rho}{2\sqrt{e^{f(x,\rho)}(1 + r'^2)}},$$

$$\partial_x \left(\frac{r'}{\sqrt{e^{f(x,\rho)}(1 + r'^2)}} \right) = \frac{2r'' - f_x r'(1 + r'^2)}{2\sqrt{e^{f(x,\rho)}(1 + r'^2)}^3}, \ \partial_\rho \left(\frac{r'}{\sqrt{e^{f(x,\rho)}(1 + r'^2)}} \right) = \frac{-f_\rho r'}{2\sqrt{e^{f(x,\rho)}(1 + r'^2)}},$$

Hence, by substituting into the formula for $\nabla_{e_1}^{e_1}$ the expressions found for $e_1(E^1)$ and $e_1(E^2)$, as well as those for the corresponding Christoffel symbols, we finally obtain:

$$\nabla_{e_1}^{e_1} = \left[\frac{r' f_\rho - r'^2 f_x}{2e^{f(x,\rho)}(1 + r'^2)} - \frac{r'r''}{e^{f(x,\rho)}(1 + r'^2)^2} \right] \partial_x + \left[\frac{r' f_x - f_\rho}{2e^{f(x,\rho)}(1 + r'^2)} + \frac{r''}{e^{f(x,\rho)}(1 + r'^2)^2} \right] \partial_\rho,$$

Let us proceed in the same manner for the other covariant derivative we need:

$$\nabla_{e_2}^{e_2} = -\frac{f_x}{2e^{f(x,\rho)}} \partial_x - \frac{2\rho + \rho^2 f_\rho}{2\rho^2 e^{f(x,\rho)}} \partial_\rho.$$

Since $e_2(F^i) = 0$ for $i = 1, 2, 3$.

For the scalar products we obtain:

$$g\left(\nabla_{e_1}^{e_1}, v\right) = \frac{f_\rho - r' f_x}{2\sqrt{e^{f(x,\rho)}\left(1+r'^2\right)}} - \frac{r''}{\sqrt{e^{f(x,\rho)}\left(1+r'^2\right)^3}},$$

and

$$g\left(\nabla_{e_2}^{e_2}, v\right) = \frac{-\rho r' f_x + 2 + \rho f_\rho}{2\rho\sqrt{e^{f(x,\rho)}\left(1+r'^2\right)}}.$$

Hence, by multiplying the two terms and performing the obvious semplifications, the formula for the mean curvature H_g becomes [12]:

$$H_g = \frac{1}{2}[g\left(\nabla_{e_1}^{e_1}, v\right) + g\left(\nabla_{e_2}^{e_2}, v\right)] = \frac{\left(-r'\rho f_x + 1 + \rho f_\rho\right)\left(1+r'^2\right) - r''\rho}{2\rho\sqrt{e^{f(x,\rho)}}\sqrt[2]{\left(1+r'^2\right)^3}}.$$

5 Proof of the Theorem

Since the last formula gives the mean curvature of the surface S as a function of its generating curve $r(x)$ and of $f(x,\rho)$, we can use it to solve H_g=c (constant) for the function $f(x,\rho)$. To do that it is convenient to introduce the following change of variable:

$$t = \rho - r(x),$$

So that the curve $\rho = r(x)$ is mapped to the line $t = 0$. Hence in the new coordinates $(\tilde{x}, t) = (x, \rho - r(x))$ the partial derivatives of f are:

$$f_\rho = f_t, f_x = f_{\tilde{x}} - r' f_t,$$

and the mean curvature H_g is:

$$H_g(x, r(x)) = H_g(\tilde{x}, 0) = \frac{\left(-r' r\left(f_{\tilde{x}} - r' f_t\right) + 1 + r f_t\right)\left(1+r'^2\right) - r r''}{2r\sqrt{e^{f(x,r)}}\sqrt[2]{\left(1+r'^2\right)^3}}.$$

A priori f is the most general function of two variables, but to simplify the computations we restrict our attention to those functions which vanish on S. We now choose any extension of f_t for $t > 0$, compatible with the condition $f\left(\tilde{x}, 0\right) = 0, f_{\tilde{x}} = 0, f = 0$, that gives.

$$H_g = \frac{(1 + r'^2 - r r'') + r f_t (1+r'^2)^2}{2r\sqrt[2]{\left(1+r'^2\right)^3}} = H_\varepsilon + \frac{f_t\sqrt{1+r'^2}}{2}.$$

where H_ε is the mean curvature of S in the euclidean metric.

6 Remarks

(1) If we choose $f\left(\tilde{x},t\right)=k$ a constant we obtain [13].

$$H_{g}=\frac{H_{\varepsilon}}{\sqrt{1+e^{k}}}.$$

which is the known scaling formula for the mean curvature under homothety.

(2) If $S=S(R)$ is the sphere of radius R:

$$r=\sqrt{R^2-x^2},1+r'^2=\frac{R^2}{R^2-x^2},r''=\frac{-R^2}{\left(R^2-x^2\right)^{\frac{3}{2}}},H_{\varepsilon}=\frac{1}{R}.$$

hence

$$f\left(x,t\right)=f\left(\tilde{x},t\right)=\frac{2(R-1)\sqrt{R^2-x^2}}{R^2}t.$$

In this way we immerse $S\left(R\right)$ in (R^3,g) for $g=e^{f(x,\rho)}\varepsilon$ a conformally flat metric which is not obtained as homothetic expansion of the euclidean one.

7 Conclusion

An algorithm of rotation surfaces with given principal curvature function is presented. The vector of the rotation surfaces is obtained by solving a second-order differential equation with proper initial condition, so we can get the rotation surfaces which satisfied the conditions. Some practical examples are given to indicate the algorithm is feasible and is carried out easily. A new method for engineering design and surface modeling of rotation surfaces is presented.

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Competing Interests

Authors have declared that no competing interests exist.

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