



## The Kannan's Fixed Point Theorem in a Cone Hexagonal Metric Spaces

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### Authors' contributions

This work was carried out in collaboration between both authors. Author AA proposed the main idea of this paper, performed all the steps of proof and wrote the first draft of the manuscript. Author EH managed the analysis of the research work and literature searches. Both authors read and approved the final manuscript.

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## ABSTRACT

In this paper, we prove Kannan's fixed point theorem in a cone hexagonal metric space. Our result extend and improve the recent result of Jleli and Samet [The Kannan's fixed point theorem in a cone rectangular metric space, J. Nonlinear Sci. Appl., 2 (2009), 161 - 167], and many existing results in the literature. Example is given showing that our result are proper extensions of the existing ones.

*Keywords:* Cone hexagonal metric space; fixed point; iterative sequence; Kannan's mapping.

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## 1 INTRODUCTION

In 1906, M. Fréchet [1] introduced the concept of metric spaces. A metric is a function that takes values in the set of real numbers with its usual ordering.

Let  $(X, d)$  be a metric space and  $S : X \rightarrow X$  be a mapping. Then  $S$  is called Banach contraction mapping if there exists  $\alpha \in [0, 1)$  such that

$$d(Sx, Sy) \leq \alpha d(x, y), \quad \forall x, y \in X. \quad (1.1)$$

Banach [2] proved that if  $X$  is complete, then every Banach contraction mapping has a fixed point. Thus, Banach contraction principle ensures the existence of a unique fixed point of a Banach contraction on a complete metric space. The contractive definition (1.1) implies that  $S$  is uniformly continuous. It is natural to ask if there is a contractive definition which do not force  $S$  to be continuous. It was answered in affirmative by Kannan [3].

Let  $(X, d)$  be a metric space and  $S : X \rightarrow X$  be a mapping. Then  $S$  is called Kannan mapping if there exists  $\alpha \in [0, 1/2)$  such that

$$d(Sx, Sy) \leq \alpha [d(x, Sx) + d(y, Sy)], \quad \forall x, y \in X. \quad (1.2)$$

Kannan [4] proved that if  $X$  is complete, then every Kannan mapping has a fixed point. He further showed that the conditions (1.1) and (1.2) are independent of each other (see, [3, 4]). Kannan's fixed point theorem also is very important. Because Subrahmanyam [5] proved that Kannan's theorem characterizes the metric completeness. That is,  $X$  is a complete metric space if and only if every Kannan mapping on  $X$  has a fixed point.

In 2007, Huang and Zhang [6] introduced the concept of a cone metric space, they replaced the set of real numbers by an ordered Banach space and proved some fixed point theorems for contractive type conditions in cone metric spaces. Later on many authors have (for e.g., [7, 8, 9, 10]) proved fixed point theorems for different contractive types conditions in cone metric spaces.

Azam et al. [11] introduced the notion of cone rectangular metric space and proved

Banach contraction mapping principle in a cone rectangular metric space setting.

In 2009, Jleli and Samet [12] extended the Kannan's fixed point theorem in a cone rectangular metric space.

Recently, M. Garg and S. Agarwal [13] introduced the notion of cone pentagonal metric space and proved Banach contraction mapping principle in a cone pentagonal metric space.

Very recently, M. Garg [14] introduced the notion of cone hexagonal metric space and proved Banach contraction mapping principle in a cone hexagonal metric space.

Motivated by the results of [12, 14], it is our purpose in this paper to continue the study of fixed point theorem in cone hexagonal metric space setting. Our results improve and extend the results of Huang and Zhang [6], Jleli and Samet [12], Auwalu [15], and many others.

## 2 PRELIMINARIES

In this section, we shall give the notion of cone metric spaces and some related properties introduced in [6, 11, 13, 14], which will be needed in the sequel.

**Definition 2.1.** Let  $E$  be a real Banach space and  $P$  a subset of  $E$ .  $P$  is called a cone if and only if:

1.  $P$  is closed, nonempty, and  $P \neq \{0\}$ ;
2.  $a, b \in \mathbb{R}, a, b \geq 0$  and  $x, y \in P \implies ax + by \in P$ ;
3.  $x \in P$  and  $-x \in P \implies x = 0$ .

Given a cone  $P \subseteq E$ , we defined a partial ordering  $\leq$  with respect to  $P$  by  $x \leq y$  if and only if  $y - x \in P$ . We shall write  $x < y$  to indicate that  $x \leq y$  but  $x \neq y$ , while  $x \ll y$  will stand for  $y - x \in \text{int}(P)$ , where  $\text{int}(P)$  denotes the interior of  $P$ .

**Definition 2.2.** A cone  $P$  is called normal if there is a number  $k > 0$  such that for all  $x, y \in E$ , the inequality

$$0 \leq x \leq y \implies \|x\| \leq k\|y\|. \quad (2.1)$$

The least positive number  $k$  satisfying (2.1) is called the normal constant of  $P$ .

In this paper, we always suppose that  $E$  is a real Banach space and  $P$  is a cone in  $E$  with  $\text{int}(P) \neq \emptyset$  and  $\leq$  is a partial ordering with respect to  $P$ .

**Definition 2.3.** Let  $X$  be a nonempty set. Suppose that the mapping  $\rho : X \times X \rightarrow E$  satisfies:

1.  $0 < \rho(x, y)$  for all  $x, y \in X$  and  $\rho(x, y) = 0$  if and only if  $x = y$ ;
2.  $\rho(x, y) = \rho(y, x)$  for all  $x, y \in X$ ;
3.  $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$  for all  $x, y, z \in X$ ;

Then  $\rho$  is called a cone metric on  $X$ , and  $(X, \rho)$  is called a cone metric space.

*Remark 2.1.* Every metric space is a cone metric space. The converse is not necessarily true (e.g., see [6]).

**Definition 2.4.** Let  $X$  be a nonempty set. Suppose that the mapping  $\rho : X \times X \rightarrow E$  satisfies:

1.  $0 < \rho(x, y)$  for all  $x, y \in X$  and  $\rho(x, y) = 0$  if and only if  $x = y$ ;
2.  $\rho(x, y) = \rho(y, x)$  for all  $x, y \in X$ ;
3.  $\rho(x, y) \leq \rho(x, w) + \rho(w, z) + \rho(z, y)$  for all  $x, y \in X$  and for all distinct points  $w, z \in X - \{x, y\}$  [rectangular property].

Then  $\rho$  is called a cone rectangular metric on  $X$ , and  $(X, \rho)$  is called a cone rectangular metric space.

*Remark 2.2.* Every cone metric space and so metric space is cone rectangular metric space. The converse is not necessarily true (e.g., see [11]).

**Definition 2.5.** Let  $X$  be a nonempty set. Suppose that the mapping  $d : X \times X \rightarrow E$  satisfies:

1.  $0 < d(x, y)$  for all  $x, y \in X$  and  $d(x, y) = 0$  if and only if  $x = y$ ;
2.  $d(x, y) = d(y, x)$  for  $x, y \in X$ ;
3.  $d(x, y) \leq d(x, z) + d(z, w) + d(w, u) + d(u, y)$  for all  $x, y, z, w, u \in X$  and for all distinct points  $z, w, u \in X - \{x, y\}$  [pentagonal property].

Then  $d$  is called a cone pentagonal metric on  $X$ , and  $(X, d)$  is called a cone pentagonal metric space.

*Remark 2.3.* Every cone rectangular metric space and so cone metric space is cone pentagonal metric space. The converse is not necessarily true (e.g., see [13]).

**Definition 2.6.** Let  $X$  be a nonempty set. Suppose that the mapping  $d : X \times X \rightarrow E$  satisfies:

1.  $0 < d(x, y)$  for all  $x, y \in X$  and  $d(x, y) = 0$  if and only if  $x = y$ ;
2.  $d(x, y) = d(y, x)$  for  $x, y \in X$ ;
3.  $d(x, y) \leq d(x, z) + d(z, w) + d(w, u) + d(u, v) + d(v, y)$  for all  $x, y, z, w, u, v \in X$  and for all distinct points  $z, w, u, v \in X - \{x, y\}$  [hexagonal property].

Then  $d$  is called a cone hexagonal metric on  $X$ , and  $(X, d)$  is called a cone hexagonal metric space.

*Remark 2.4.* Every cone pentagonal metric space and so cone rectangular metric space is cone hexagonal metric space. The converse is not necessarily true (e.g., see [14]).

**Definition 2.7.** Let  $(X, d)$  be a cone hexagonal metric space. Let  $\{x_n\}$  be a sequence in  $X$  and  $x \in X$ . If for every  $c \in E$  with  $0 \ll c$  there exist  $n_0 \in \mathbb{N}$  and that for all  $n > n_0$ ,  $d(x_n, x) \ll c$ , then  $\{x_n\}$  is said to be convergent and  $\{x_n\}$  converges to  $x$ , and  $x$  is the limit of  $\{x_n\}$ . It is denoted by  $\lim_{n \rightarrow \infty} x_n = x$  or  $x_n \rightarrow x$  as  $n \rightarrow \infty$ .

**Definition 2.8.** Let  $(X, d)$  be a cone hexagonal metric space. Let  $\{x_n\}$  be a sequence in  $X$ . If for every  $c \in E$ , with  $0 \ll c$  there exist  $n_0 \in \mathbb{N}$  such that for all  $n, m > n_0$ ,  $d(x_n, x_m) \ll c$ , then  $\{x_n\}$  is called Cauchy sequence in  $X$ .

**Definition 2.9.** Let  $(X, d)$  be a cone hexagonal metric space. If every Cauchy sequence is convergent in  $(X, d)$ , then  $X$  is called a complete cone hexagonal metric space.

**Lemma 2.1.** Let  $(X, d)$  be a cone hexagonal metric space and  $P$  be a normal cone with normal constant  $k$ . Let  $\{x_n\}$  be a sequence in  $X$ , then  $\{x_n\}$  converges to  $x$  if and only if  $\|d(x_n, x)\| \rightarrow 0$  as  $n \rightarrow \infty$ .

**Lemma 2.2.** Let  $(X, d)$  be a cone hexagonal metric space and  $P$  be a normal cone with normal constant  $k$ . Let  $\{x_n\}$  be a sequence in  $X$ , then  $\{x_n\}$  is a Cauchy sequence if and only if  $\|d(x_n, x_{n+m})\| \rightarrow 0$  as  $n \rightarrow \infty$ .

In fact, for a cone hexagonal (or rectangular) metric space, the uniqueness of the limit of a sequence does not hold (see [12]). However, the limit is unique for a convergent Cauchy sequence as shown below.

**Lemma 2.3.** Let  $(X, d)$  be a complete cone hexagonal metric space,  $P$  be a normal cone with normal constant  $k$ . Let  $\{x_n\}$  be a Cauchy sequence in  $X$  and suppose that there is natural number  $N$  such that:

1.  $x_n \neq x_m$  for all  $n, m > N$ ;
2.  $x_n, x$  are distinct points in  $X$  for all  $n > N$ ;
3.  $x_n, y$  are distinct points in  $X$  for all  $n > N$ ;
4.  $x_n \rightarrow x$  and  $x_n \rightarrow y$  as  $n \rightarrow \infty$ .

Then  $x = y$ .

*Proof.* The proof is similar to the proof of ([12]-Lemma 1.10). □

### 3 MAIN RESULTS

In this section, we prove Kannan fixed point theorem in a cone hexagonal metric space. Our result generalized the well known Kannan's fixed point theorem [4], and results of Jleli and Samet [12], Auwalu [15], and others.

**Theorem 3.1.** Let  $(X, d)$  be a complete cone hexagonal metric space,  $P$  be a normal cone with normal constant  $k$ . Suppose a mapping  $S : X \rightarrow X$  satisfy the contractive condition:

$$d(Sx, Sy) \leq \lambda[d(Sx, x) + d(Sy, y)], \quad (3.1)$$

for all  $x, y \in X$ , where  $\lambda \in [0, 1/2)$ . Then

1.  $S$  has a unique fixed point in  $X$ .
2. For any  $x \in X$ , the iterative sequence  $\{S^n x\}$  converges to the fixed point.

*Proof.* Let  $x \in X$ . From (3.1), we have

$$d(Sx, S^2x) \leq \lambda[d(x, Sx) + d(Sx, S^2x)],$$

i.e,

$$d(Sx, S^2x) \leq \frac{\lambda}{1-\lambda}d(x, Sx).$$

Again

$$d(S^2x, S^3x) \leq \lambda[d(Sx, S^2x) + d(S^2x, S^3x)],$$

i.e,

$$d(S^2x, S^3x) \leq \frac{\lambda}{1-\lambda}d(Sx, S^2x) \leq \left(\frac{\lambda}{1-\lambda}\right)^2 d(x, Sx).$$

Thus, in general, if  $n$  is a positive integer, then

$$\begin{aligned} d(S^n x, S^{n+1} x) &\leq \left(\frac{\lambda}{1-\lambda}\right)^n d(x, Sx) \\ &= r^n d(x, Sx), \end{aligned} \quad (3.2)$$

where  $r = \left(\frac{\lambda}{1-\lambda}\right) \in [0, 1)$ .

We divide the proof into two cases.

**First case :**

Let  $S^m x = S^n x$  for some  $m, n \in \mathbb{N}$ ,  $m \neq n$ . Let  $m > n$ . Then  $S^{m-n}(S^n x) = S^n x$ , i.e.  $S^p y = y$ , where  $p = m - n$ ,  $y = S^n x$ . Now since  $p > 1$ , we have

$$\begin{aligned} d(y, Sy) &= d(S^p y, S^{p+1} y) \\ &\leq r^n d(y, Sy). \end{aligned}$$

Since  $r \in [0, 1)$ , we obtain  $-d(y, Sy) \in P$  and  $d(y, Sy) \in P$ , which implies that

$$\|(y, Sy)\| = 0.$$

That is,  $Sy = y$ .

**Second case :**

Assume that  $S^m x \neq S^n x$  for all  $m, n \in \mathbb{N}$ ,  $m \neq n$ . From (3.2), and the fact that  $0 \leq \lambda < r < 1$ , we have

$$\begin{aligned} d(S^n x, S^{n+1} x) &\leq r^n d(x, Sx) \\ &\leq r^n d(x, Sx) + r^{n+1} d(x, Sx) + r^{n+2} d(x, Sx) \\ &\leq r^n (1 + r + r^2 + \dots) d(x, Sx) \\ &\leq \frac{r^n}{1-r} d(x, Sx), \end{aligned}$$

$$\begin{aligned} d(S^n x, S^{n+2} x) &\leq \lambda [d(S^{n-1} x, S^n x) + d(S^{n+1} x, S^{n+2} x)] \\ &\leq \lambda [r^{n-1} d(x, Sx) + r^{n+1} d(x, Sx)] \\ &\leq r^n d(x, Sx) + r^{n+1} d(x, Sx) + r^{n+2} d(x, Sx) \\ &\leq r^n (1 + r + r^2 + \dots) d(x, Sx) \\ &= \frac{r^n}{1-r} d(x, Sx), \end{aligned}$$

$$\begin{aligned} d(S^n x, S^{n+3} x) &\leq \lambda [d(S^{n-1} x, S^n x) + d(S^{n+2} x, S^{n+3} x)] \\ &\leq \lambda [r^{n-1} d(x, Sx) + r^{n+2} d(x, Sx)] \\ &\leq r^n d(x, Sx) + r^{n+1} d(x, Sx) + r^{n+2} d(x, Sx) \\ &\leq r^n (1 + r + r^2 + \dots) d(x, Sx) \\ &= \frac{r^n}{1-r} d(x, Sx). \end{aligned}$$

and

$$\begin{aligned} d(S^n x, S^{n+4} x) &\leq \lambda [d(S^{n-1} x, S^n x) + d(S^{n+3} x, S^{n+4} x)] \\ &\leq \lambda [r^{n-1} d(x, Sx) + r^{n+3} d(x, Sx)] \\ &\leq r^n d(x, Sx) + r^{n+1} d(x, Sx) + r^{n+2} d(x, Sx) + r^{n+3} d(x, Sx) \\ &\leq r^n (1 + r + r^2 + r^3 + \dots) d(x, Sx) \\ &= \frac{r^n}{1-r} d(x, Sx). \end{aligned}$$

Now, if  $m > 4$  and  $m := 4k + 1$ ,  $k \geq 1$  and using the fact that  $S^p x \neq S^q x$  for  $p, q \in \mathbb{N}$ ,  $p \neq q$ , by

hexagonal property, we obtain

$$\begin{aligned}
 d(S^n x, S^{n+4k+1} x) &\leq d(S^{n+4k+1} x, S^{n+4k} x) + d(S^{n+4k} x, S^{n+4k-1} x) + d(S^{n+4k-1} x, S^{n+4k-2} x) \\
 &\quad + d(S^{n+4k-2} x, S^{n+4k-3} x) + d(S^{n+4k-3} x, S^n x) \\
 &\leq d(S^{n+4k} x, S^{n+4k+1} x) + d(S^{n+4k-1} x, S^{n+4k} x) + d(S^{n+4k-2} x, S^{n+4k-1} x) \\
 &\quad + d(S^{n+4k-2} x, S^{n+4k-3} x) + d(S^{n+4k-3} x, S^{n+4k-4} x) + \dots \\
 &\quad + d(S^{n+2} x, S^{n+1} x) + d(S^{n+1} x, S^n x) \\
 &= d(S^n x, S^{n+1} x) + d(S^{n+1} x, S^{n+2} x) + \dots + d(S^{n+4k-1} x, S^{n+4k} x) \\
 &\quad + d(S^{n+4k} x, S^{n+4k+1} x) \\
 &\leq r^n d(x, Sx) + r^{n+1} d(x, Sx) + \dots + r^{n+4k-1} d(x, Sx) + r^{n+4k} d(x, Sx) \\
 &\leq r^n (1 + r + r^2 + \dots) d(x, Sx) \\
 &= \frac{r^n}{1-r} d(x, Sx).
 \end{aligned}$$

Similarly, if  $m > 5$  and  $m := 4k + 2$ ,  $k \geq 1$  and using the fact that  $S^p x \neq S^q x$  for  $p, q \in \mathbb{N}$ ,  $p \neq q$ , by hexagonal property, we obtain

$$d(S^n x, S^{n+4k+2} x) \leq \frac{r^n}{1-r} d(x, Sx).$$

Also, if  $m > 6$  and  $m := 4k + 3$ ,  $k \geq 1$  and using the fact that  $S^p x \neq S^q x$  for  $p, q \in \mathbb{N}$ ,  $p \neq q$ , by hexagonal property, we obtain

$$d(S^n x, S^{n+4k+3} x) \leq \frac{r^n}{1-r} d(x, Sx).$$

Also, if  $m > 7$  and  $m := 4k + 4$ ,  $k \geq 1$  and using the fact that  $S^p x \neq S^q x$  for  $p, q \in \mathbb{N}$ ,  $p \neq q$ , by hexagonal property, we obtain

$$d(S^n x, S^{n+4k+4} x) \leq \frac{r^n}{1-r} d(x, Sx).$$

Thus, combining the above cases, we have

$$d(S^n x, S^{n+m} x) \leq \frac{r^n}{1-r} d(x, Sx), \quad \forall m, n \in \mathbb{N}.$$

Since  $P$  is a normal cone with normal constant  $k$ , therefore, by (2.1), we have

$$\|d(S^n x, S^{n+m} x)\| \leq \frac{kr^n}{1-r} \|d(x, Sx)\|, \quad \forall m, n \in \mathbb{N}.$$

Since

$$\lim_{n \rightarrow \infty} \frac{kr^n}{1-r} \|d(x, Sx)\| = 0,$$

we have that

$$\lim_{n \rightarrow \infty} \|d(S^n x, S^{n+m} x)\| = 0, \quad \forall m, n \in \mathbb{N}. \tag{3.3}$$

Therefore, by Lemma 2.2,  $\{S^n x\}$  is a cauchy sequence in  $X$ . By completeness of  $X$ , there exists a point  $z \in X$  such that

$$\lim_{n \rightarrow \infty} S^n x = z. \tag{3.4}$$

We shall now show that  $z$  is a fixed point of  $S$ . i.e.,  $Sz = z$ . Without loss of generality, we can assume that  $S^r x \neq z, Sz$  for any  $r \in \mathbb{N}$ . By hexagonal property, we have

$$\begin{aligned} d(z, Sz) &\leq d(z, S^n x) + d(S^n x, S^{n+1} x) + d(S^{n+1} x, S^{n+2} x) + d(S^{n+2} x, S^{n+3} x) + d(S^{n+3} x, Sz) \\ &\leq d(z, S^n x) + d(S^n x, S^{n+1} x) + d(S^{n+1} x, S^{n+2} x) + d(S^{n+2} x, S^{n+3} x) + \\ &\quad \lambda[d(S^{n+2} x, S^{n+3} x) + d(z, Sz)], \end{aligned}$$

which implies that

$$d(z, Sz) \leq \frac{1}{1-\lambda} [d(z, S^n x) + d(S^n x, S^{n+1} x) + d(S^{n+1} x, S^{n+2} x) + (1+\lambda)d(S^{n+2} x, S^{n+3} x)].$$

Hence,

$$\|d(z, Sz)\| \leq \frac{k}{1-\lambda} [\|d(z, S^n x)\| + \|d(S^n x, S^{n+1} x)\| + \|d(S^{n+1} x, S^{n+2} x)\| + (1+\lambda)\|d(S^{n+2} x, S^{n+3} x)\|].$$

Letting  $n \rightarrow \infty$ , we have  $\|d(z, Sz)\| = 0$ . Hence,  $Sz = z$ . i.e.,  $z$  is a fixed point of  $S$ .

Now, we show that  $z$  is unique. Suppose  $z'$  is another fixed point of  $S$ , that is  $Sz' = z'$ . Therefore,

$$d(z, z') = d(Sz, Sz') \leq \lambda[d(z, Sz) + d(z', Sz')] = 0,$$

which implies that

$$d(z, z') = 0.$$

That is,  $z = z'$ . This completes the proof of the theorem. □

## 4 CONCLUSIONS

**Corollary 4.1.** (see [15]) *Let  $(X, d)$  be a complete cone pentagonal metric space and  $P$  be a normal cone with normal constant  $k$ . Suppose the mapping  $S : X \rightarrow X$  satisfy the contractive condition:*

$$d(Sx, Sy) \leq \lambda[d(x, Sx) + d(y, Sy)], \quad (4.1)$$

for all  $x, y \in X$ , where  $\lambda \in [0, 1/2)$ . Then

1.  $S$  has a unique fixed point in  $X$ .
2. For any  $x \in X$ , the iterative sequence  $\{S^n x\}$  converges to the fixed point.

*Proof.* This follows from the Remark 2.4 and Theorem 3.1. □

**Corollary 4.2.** (see [12]) *Let  $(X, d)$  be a complete cone rectangular metric space and  $P$  be a normal cone with normal constant  $k$ . Suppose the mapping  $S : X \rightarrow X$  satisfy the contractive condition:*

$$d(Sx, Sy) \leq \lambda[d(Sx, x) + d(Sy, y)], \quad (4.2)$$

for all  $x, y \in X$ , where  $\lambda \in [0, 1/2)$ . Then

1.  $S$  has a unique fixed point in  $X$ .
2. For any  $x \in X$ , the iterative sequence  $\{S^n x\}$  converges to the fixed point.

*Proof.* This follows from the Remark 2.3 and Corollary 4.1. □

**Corollary 4.3.** (see [6]) *Let  $(X, d)$  be a complete cone metric space and  $P$  be a normal cone with normal constant  $k$ . Suppose the mapping  $S : X \rightarrow X$  satisfy the contractive condition:*

$$d(Sx, Sy) \leq \lambda[d(Sx, x) + d(Sy, y)], \quad (4.3)$$

for all  $x, y \in X$ , where  $\lambda \in [0, 1/2)$ . Then

1.  $S$  has a unique fixed point in  $X$ .
2. For any  $x \in X$ , the iterative sequence  $\{S^n x\}$  converges to the fixed point.

*Proof.* This follows from the Remark 2.2 and Corollary 4.2. □

To illustrate Theorem 3.1, we give the following example.

**Example 4.4.** Let  $X = \{1, 2, 3, 4, 5, 6\}$ ,  $E = \mathbb{R}^2$  and  $P = \{(x, y) : x, y \geq 0\}$  is a normal cone in  $E$ . Define  $d : X \times X \rightarrow E$  as follows:

$$\begin{aligned} d(x, x) &= 0, \forall x \in X; \\ d(1, 2) &= d(2, 1) = (5, 10); \\ d(1, 3) &= d(3, 1) = d(1, 4) = d(4, 1) = d(1, 5) = d(5, 1) = d(2, 3) = d(3, 2) = d(2, 4) = d(4, 2) \\ &= d(2, 5) = d(5, 2) = d(3, 4) = d(4, 3) = d(3, 5) = d(5, 3) = d(4, 5) = d(5, 4) = (1, 2); \\ d(1, 6) &= d(6, 1) = d(2, 6) = d(6, 2) = d(3, 6) = d(6, 3) = d(4, 6) = d(6, 4) = d(5, 6) = d(6, 5) = (4, 8). \end{aligned}$$

Then  $(X, d)$  is a complete cone hexagonal metric space, but  $(X, d)$  is not a complete cone pentagonal metric space because it lacks the pentagonal property:

$$\begin{aligned} (5, 10) &= d(1, 2) > d(1, 3) + d(3, 4) + d(4, 5) + d(5, 2) \\ &= (1, 2) + (1, 2) + (1, 2) + (1, 2) \\ &= (4, 8), \text{ as } (5, 10) - (4, 8) = (1, 2) \in P. \end{aligned}$$

Now, we define a mapping  $S : X \rightarrow X$  as follows

$$S(x) = \begin{cases} 5, & \text{if } x \neq 6; \\ 2, & \text{if } x = 6. \end{cases}$$

Observe that

$$\begin{aligned} d(S(1), S(2)) &= d(S(1), S(3)) = d(S(1), S(4)) = d(S(1), S(5)) = d(S(2), S(3)) \\ &= d(S(2), S(4)) = d(S(2), S(5)) = d(S(3), S(4)) = d(S(3), S(5)) = 0. \end{aligned}$$

And in all other cases  $d(S(x), S(y)) = (1, 2)$ ,  $d(x, y) = (4, 8)$ .

We remark that  $S$  is not a contractive mapping with respect to the standard metric in  $X$ , because we have

$$|S6 - S3| = 3 = |6 - 3|.$$

However,  $S$  satisfies

$$d(Sx, Sy) \leq \lambda [d(x, Sx) + d(y, Sy)], \quad \forall x, y \in X,$$

with  $\lambda = 1/4$ . Applying Theorem 3.1, we obtain that  $S$  admits a unique fixed point, that is  $z = 5$ .

In the above Example, results of Auwalu [15], Jleli and Samet [12], or Huang and Zhang [6] are not applicable to obtain the fixed point of the mapping  $S$  on  $X$ . Since  $(X, d)$  is not a cone pentagonal, or cone rectangular, or cone metric space.

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## COMPETING INTERESTS

Authors have declared that no competing interests exist.

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