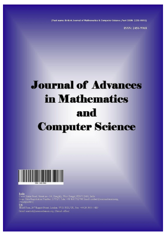

Journal of Advances in Mathematics and Computer Science

37(10): 90-104, 2022; Article no.JAMCS.93804

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

The Solution of Fractional Diffusion-reaction Equation Via the Regular Perturbation Method (RPM)

Bationo Jérémie Yiyuréboula [a*], Yaya Moussa [a] and Bassono Francis [a]

[a] *University Joseph Ki Zerbo, Burkina Faso.*

Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JAMCS/2022/v37i101718

Open Peer Review History:

This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/93804

Received: 20 September 2022
Accepted: 22 November 2022
Published: 25 November 2022

Original Research Article

Abstract

In this paper, we implement Regular Perturbation Method (RPM) of the Solving fractional diffusion-reaction equation, in order to determine the exact analytical solutions of some linear fractional diffusion-reaction equation. In general, the solving using this method allow to obtain exact or approximate solutions. For the case of the diffusion and diffusion-convection equations solved in this document, the solutions obtained are exact. By comparing these solutions with those obtained by other researchers using other methods for a certain value of the parameter α, we obtain the same results.

Keywords: Linear fractional diferential equation; regular perturbation method; Mittag-Leffler; Caputo fractional derivative or integral.

2010 Mathematics Subject Classification: 05A17, 11B73, 05A18.

1 Introduction

Today, the notion of fractional calculus is essential in the resolution of partial differential equations. Indeed, several researchers have applied it to several methods such as the method of Adomian, the Homotopy Perturbation

**Corresponding author: E-mail: jeremiebationo@yahoo.fr;*

method (HPM), etc. However its application remains partial and fragmented. In this document, we will attempt to apply it using the Regular Perturbations method for solving linear fractional diffusion-convection equations. The objective of this work is to find solutions to fractional equations using the Regular Perturbations method for $0 < \alpha \leq 1$. Secondly, we will compare the solutions obtained with the solutions resulting from another resolution method in a previous search for a given value of α. α is defined in the problem below. Searchers such as Y. MINOUGOU have already solved the case where $\alpha = 1$ with several methods [1, 2, 3, 4, 5].

Let be the following fractional equation (P):

$$(P)\begin{cases} \dfrac{\partial^{\alpha} u\,(x,t)}{\partial t^{\alpha}} = f_{\varepsilon}(t, u(x,t)) \;\; \text{where } 0 < \alpha \leq 1; \\ u(x,0) = g(x) \end{cases}$$

$(x;t) \in \Omega = \mathbb{R} \times [0, +\infty[$ and $u\,(x,t) \in L^2\,(\Omega)$
where g is any function dependent only on x, and f_{ε} is a continuous function. f_{ε} can be the right-hand member of a fractional diffusion- convection equation for example, where ε is the pertubation coefficient, with $0 < \varepsilon \ll 1$.

2 Preliminaries

In this part, we will recall some very important notions that come into play imperatively in fractional calculus. These are the notions of gamma functions, Beta and Mitag Leffler function as well as some notions of convergence and of solution uniqueness.

2.1 Gamma, beta and mittag leffler functions

2.1.1 Definition 1 [6]

The Gamma function is a function on]0;1[, defined by the following integral:

$\Gamma(s) = \int_0^{+\infty} e^{-t} t^{s-1} dt; \quad s \in \mathbb{C}$ and $Re(s) > 0$. Thus: $\Gamma(1) = 1$;
$\Gamma(s)$ is a monotonous and strictly decreasing function for $0 \leq s \leq 1$.

2.1.2 Property 1:

$$\cdot \forall s > 0,\; \Gamma(s+1) = s\Gamma(s);\; \Gamma(\frac{1}{2}) = \sqrt{\pi}.$$
$$\cdot \forall n \in \mathbb{N},\; \Gamma(n+1) = n!;\; \text{with } 0! = \; 1.$$

2.1.3 Definition 2 [6]

The Beta function is the function defined by:

$$\beta(u,v) = \int_0^1 (1-z)^{u-1} z^{v-1} dz;$$

where $Re(u) > 0$ and $Re(v) > 0$.

2.1.4 Property 2:

$\forall u \in \mathbb{C}$; $\forall v \in \mathbb{C}$ where $Re(u) > 0$ and $Re(v) > 0$,

Thus:

$$\beta(u,v) = \beta(v,u) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$$

2.1.5 Definition 3 [7], [8]

For $s \in \mathbb{C}$, the Mittag-Leffler function denoted $E_a(s)$ with $\alpha > 0$ is defined by:

$$E_a(s) = \sum_{k=0}^{+\infty} \frac{s^k}{\Gamma(k\alpha+1)};$$

when it depends on a single parameter α.

2.1.6 Property 3:

$\forall s \in \mathbb{C}$ where $Re(s) > 0$, we have:

$E_1(s) = e^s, E_2(s) = ch(\sqrt{s})$ where ch denotes the hyberbolic cosine.

2.1.7 Property 4:

$\forall s \in \mathbb{C}$ where $Re(s) > 0$, the Mittag-Leffler function is indefinitely derivable.

We have:

$$E_a^{(m)}(s) = \sum_{k=0}^{+\infty} k(k-1)(k-2)...(k-m+1)\frac{s^{k-m}}{\Gamma(k\alpha+1)} \text{ with } m \in \mathbb{N};$$

where $E_a^{(m)}$ is the m-th derivative of E_a dependent on a single parameter α.

2.1.8 Definition 4 [6]

Let considere $f \in L^1([0,T]), T > 0$. The Riemann-Liouville fractional integral of the function f of order $\alpha \in \mathbb{C}, (Re(\alpha) > 0)$ noted I_α is defined by:

$$I_\alpha f(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(x,\tau)d\tau; t > 0; x \in \mathbb{R}.$$

2.1.9 Definition 5 [9], [10], [6]

Let considere $f \in L^1([0,T]), T > 0$ a integrable function on $[0,T]$. The fractional derivative in the sens of Riemann-Liouville of the function f of order $\alpha \in \mathbb{C}, (Re(\alpha) > 0)$ noted $D_\alpha f$ is defined by:

$$D_\alpha f(x,t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} f(x,\tau)d\tau; t > 0; x \in \mathbb{R}.$$

2.2 Concept of convergence and uniqueness of the solution

Let's consider the problem (P) defined by:

$$(P)\begin{cases} \dfrac{\partial^\alpha u\,(x,t)}{\partial t^\alpha} = f_\varepsilon(t,u(x,t)) \ \text{ where } 0<\alpha\leq 1 \text{ and } 0<\varepsilon \ll 1; \\ u(x,0)=g(x) \end{cases}$$

with its solution in the form:

$$u(x;t)=\sum_{n=0}^{+\infty}\varepsilon^n u_n(x,t),$$

where $g(x)\in C(\Omega)$.
we obtain:

$$\sum_{n=0}^{+\infty}\varepsilon^n u_n(x,t)=\sum_{n=0}^{+\infty}\varepsilon^n u_n(x,0)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau,\sum_{n=0}^{+\infty}\varepsilon^n u_n(x,\tau))d\tau$$

As f is linear, we obtain:

$$\sum_{n=0}^{+\infty}\varepsilon^n u_n(x,t)=\sum_{n=0}^{+\infty}\varepsilon^n u_n(x,0)+\sum_{n=0}^{+\infty}\varepsilon^n\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau,u_n(x,\tau))d\tau.$$

2.2.1 Proposition [10], [6], [7], [8]

Let's suppose (P) is a linear diffusion-reaction equation, where $u(x,t)\in C(\Omega)$ and $f=\dfrac{\partial^2 u}{\partial x^2}+\lambda\dfrac{\partial u}{\partial x}+\gamma u\in C^2(\Omega)$,with $\Omega=\mathbb{R}\times[0;t]$ and $0\leq\tau\leq t<T<+\infty$. If $\exists\ M=\sup\limits_{x,\tau\in\Omega}\left|\dfrac{\partial^2 u(x,\tau)}{\partial x^2}\right|, N=\sup\limits_{x,\tau\in\Omega}\left|\dfrac{\partial u(x,\tau)}{\partial x}\right|$ and $m=\sup\limits_{x,\tau\in\Omega}|u(x,\tau)|$ such as: $\forall x,\tau\in\Omega, \left|\dfrac{\partial^2 u(x,\tau)}{\partial x^2}\right|\leq M, \left|\dfrac{\partial u(x,\tau)}{\partial x}\right|\leq N$ and $|u(x,\tau)|\leq m, n\geq 0$, then the series $\left(\sum\limits_{n=0}^{+\infty}\varepsilon^n u_n(x,t)\right)$ is convergent and the solution of the equation (P) exist and is unique.

2.2.2 Proof

Let's first show that the series $\left(\sum\limits_{n=0}^{+\infty}\varepsilon^n u_n(x,t)\right)$ is absolutely convergent.

Let's consider the associated algorithm below following the values of ε^n which permit to obtain:

$$\begin{cases} \varepsilon^0: & |u_0(x,t)|=\left|u(x,0)+\dfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\lambda\dfrac{\partial u_0(x,\tau)}{\partial x}+\gamma u_0(x,\tau)\right)d\tau\right| \\ \varepsilon^1: & |u_1(x,t)|=\left|\dfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\dfrac{\partial^2 u_0(x,\tau)}{\partial x^2}+\lambda\dfrac{\partial u_1(x,\tau)}{\partial x}+\gamma u_1(x,\tau)\right)d\tau\right| \\ \varepsilon^2: & |u_2(x,t)|=\left|\dfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\dfrac{\partial^2 u_1(x,\tau)}{\partial x^2}+\lambda\dfrac{\partial u_2(x,\tau)}{\partial x}+\gamma u_2(x,\tau)\right)d\tau\right| \\ \dots & \dots \\ \varepsilon^n: & |u_n(x,t)|=\left|\dfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\dfrac{\partial^2 u_{n-1}(x,\tau)}{\partial x^2}+\lambda\dfrac{\partial u_n(x,\tau)}{\partial x}+\gamma u_n(x,\tau)\right)d\tau\right| \end{cases}$$

$$\Longrightarrow \begin{cases} \varepsilon^0: & |u_0(x,t)| \leq m + \dfrac{(\lambda N+\gamma m)T^\alpha}{\Gamma(\alpha+1)} \\ \varepsilon^1: & |u_1(x,t)| \leq \dfrac{(M+\lambda N+\gamma m)T^\alpha}{\Gamma(\alpha+1)} \\ \varepsilon^2: & |u_2(x,t)| \leq \dfrac{(M+\lambda N+\gamma m)T^\alpha}{\Gamma(\alpha+1)} \\ ... & ... \\ \varepsilon^n: & |u_n(x,t)| \leq \dfrac{(M+\lambda N+\gamma m)T^\alpha}{\Gamma(\alpha+1)} \end{cases}$$

We have: $\left|\sum_{n=0}^{+\infty}\varepsilon^n u_n(x,t)\right| \leq m + \sum_{n=0}^{+\infty}\varepsilon^n\left((M+\lambda N+\gamma K)\dfrac{T^\alpha}{\Gamma(\alpha+1)}\right)$

$$\leq m + \left((M+\lambda N+\gamma K)\frac{T^\alpha}{\Gamma(\alpha+1)}\right)\sum_{n=0}^{+\infty}\varepsilon^n$$

$\Rightarrow \left|\sum_{n=0}^{+\infty}\varepsilon^n u_n(x,t)\right| \leq m + \dfrac{\left((M+\lambda N+\gamma K)\dfrac{T^\alpha}{\Gamma(\alpha+1)}\right)}{1-\varepsilon}$, because $0<\varepsilon\ll 1$

As a result, the series $\left(\sum_{n=0}^{+\infty}\varepsilon^n u_n(x,t)\right)$ is absolutely convergent therefore convergent.

Let's show now that equation (P) admits a only one solution which is written in the form:

$u(x,t) = u(x,0) + I_\alpha\left(f(\tau,u(x,\tau)\right)$

where $I_\alpha(f)$ denotes the fractional integral of f defined by:

$I_\alpha\left(f(\tau,u(x,\tau)\right) = \dfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau,u(x,\tau))d\tau$

Suppose there are two distinct solutions $u(x,t)$ and $v(x,t)$ such that:

there is $w(x,t) = u(x,t) - v(x,t) \neq 0$

Considering the following algorithms of u and v :

$$\begin{cases} u_0(x,t) = u(x,0) + \dfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\lambda\dfrac{\partial u_0(x,\tau)}{\partial x} + \gamma u_0(x,\tau)\right)d\tau \\ u_1(x,t) = \dfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\dfrac{\partial^2 u_0(x,\tau)}{\partial x^2} + \lambda\dfrac{\partial u_1(x,\tau)}{\partial x} + \gamma u_1(x,\tau)\right)d\tau \\ ... \\ u_n(x,t) = \dfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\dfrac{\partial^2 u_{n-1}(x,\tau)}{\partial x^2} + \lambda\dfrac{\partial u_n(x,\tau)}{\partial x} + \gamma u_n(x,\tau)\right)d\tau \end{cases}$$

$$\begin{cases} v_0(x,t) = v(x,0) + \dfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\lambda\dfrac{\partial v_0(x,\tau)}{\partial x} + \gamma v_0(x,\tau)\right)d\tau \\ v_1(x,t) = \dfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\dfrac{\partial^2 v_0(x,\tau)}{\partial x^2} + \lambda\dfrac{\partial v_1(x,\tau)}{\partial x} + \gamma v_1(x,\tau)\right)d\tau \\ ... \\ v_n(x,t) = \dfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left(\dfrac{\partial^2 v_{n-1}(x,\tau)}{\partial x^2} + \lambda\dfrac{\partial v_n(x,\tau)}{\partial x} + \gamma v_n(x,\tau)\right)d\tau \end{cases}$$

We have:

$$\begin{cases} w_0(x,t) = w(x,0) + \dfrac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(\lambda \dfrac{\partial w_0(x,\tau)}{\partial x} + \gamma w_0(x,\tau)\right) d\tau = 0 \\ w_1(x,t) = \dfrac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(\dfrac{\partial^2 w_0(x,\tau)}{\partial x^2} + \lambda \dfrac{\partial w_1(x,\tau)}{\partial x} + \gamma w_1(x,\tau)\right) d\tau = 0 \\ ... \\ w_n(x,t) = \dfrac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(\dfrac{\partial^2 w_{n-1}(x,\tau)}{\partial x^2} + \lambda \dfrac{\partial w_n(x,\tau)}{\partial x} + \gamma w_n(x,\tau)\right) d\tau = 0 \end{cases}$$

Because from the conditions to the limits, it happens that
$u(x,0) = g(x) = v(x,0) \Rightarrow w(x,0) = u(x,0) - v(x,0) = 0$
with $u_0(x,0) = u(x,0); v_0(x,0) = v(x,0)$ and $w_0(x,0) = w(x,0)$.
In addition, $w_0(x,t)$ is a function of $w(x,0)$ $\forall t \in [0,+\infty[$
Where from $w(x,0) = 0 \Rightarrow w_0(x,\tau) = 0$ $\forall \tau \in [0,+\infty[\Rightarrow w_0(x,t) = 0$;absurd.
$w_1(x,t)$ is too a linear function of $w(x,0) \Longrightarrow w_1(x,t) = 0$
As a result $w_n(x,t) = 0 = u_n(x,t) - v_n(x,t)$, $\forall n \geq 0$.
Thus $u(x,t) = v(x,t)$, we deduce the uniqueness of the solution of equation (P).

3 Applications

3.1 Resolution of fractional diffusion-reaction linear equation [11]

Let(P)*betheproblemtobestudieddefinedby* :

$$(P): \begin{cases} \dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \varepsilon \dfrac{\partial^2 u(x,t)}{\partial x^2} + \lambda \dfrac{\partial u(x,t)}{\partial x} + \gamma u(x,t) \text{ where } 0 < \varepsilon \ll 1; \lambda > 0; \gamma > 0 \text{ and } 0 < \alpha \leq 1 \\ u(x;0) = \sin(\omega x) \end{cases}$$

$(x,t) \in \Omega = \mathbb{R} \times [0,+\infty[$ and $u(x,t) \in L^2(\Omega)$

Finding the solution in the form: $$u(x,t) = \sum_{n=0}^{+\infty} \varepsilon^n u_n(x,t) \quad (1)$$

Let's posing: $L(u(x,t)) = \dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ and $Ru(x;t) = \varepsilon \dfrac{\partial^2 u(x,t)}{\partial x^2} + \lambda \dfrac{\partial u(x,t)}{\partial x} + \gamma u(x,t)$

By applying L^{-1} to the equation (P),we obtain:

$$u(x,t) - u(x,0) = L^{-1} Ru(x,t) \text{ with } L^{-1} = I_\alpha$$

Which equals: $$\sum_{n=0}^{+\infty} \varepsilon^n u_n(x,t) = u(x,0) + I_\alpha R\left(\sum_{n=0}^{+\infty} \varepsilon^n u_n(x,t)\right)$$

or $$\sum_{n=0}^{+\infty} \varepsilon^n u_n(x,t) = u(x,0) + I_\alpha \left(\sum_{n=0}^{+\infty} \varepsilon^{n+1} \frac{\partial^2 u_n(x,t)}{\partial x^2} + \lambda \sum_{n=0}^{+\infty} \varepsilon^n \frac{\partial u_n(x,t)}{\partial x} + \gamma \sum_{n=0}^{+\infty} \varepsilon^n u_n(x,t)\right)$$

The following equations are obtained successively:

$$\begin{cases} \varepsilon^0: & u_0(x,t) = I_\alpha\left(\lambda\dfrac{\partial u_0(x,t)}{\partial x} + \gamma u_0(x,t)\right) \\ \varepsilon^1: & u_1(x,t) = I_\alpha\left(\dfrac{\partial^2 u_0(x,t)}{\partial x^2} + \lambda\dfrac{\partial u_1(x,t)}{\partial x} + \gamma u_1(x,t)\right) \\ \varepsilon^2: & u_2(x,t) = I_\alpha\left(\dfrac{\partial^2 u_1(x,t)}{\partial x^2} + \lambda\dfrac{\partial u_2(x,t)}{\partial x} + \gamma u_2(x,t)\right) \\ \varepsilon^3: & u_3(x,t) = I_\alpha\left(\dfrac{\partial^2 u_2(x,t)}{\partial x^2} + \lambda\dfrac{\partial u_3(x,t)}{\partial x} + \gamma u_3(x,t)\right) \\ \varepsilon^4: & u_4(x,t) = I_\alpha\left(\dfrac{\partial^2 u_3(x,t)}{\partial x^2} + \lambda\dfrac{\partial u_4(x,t)}{\partial x} + \gamma u_4(x,t)\right) \\ ... & ... \\ \varepsilon^n: & u_n(x,t) = I_\alpha\left(\dfrac{\partial^2 u_{n-1}(x,t)}{\partial x^2} + \lambda\dfrac{\partial u_n(x,t)}{\partial x} + \gamma u_n(x,t)\right); n \geq 1 \end{cases}$$

This is equivalent to solving the following equations:

$$\begin{cases} \varepsilon^0: \begin{cases} \dfrac{\partial^\alpha u_0(x,t)}{\partial t^\alpha} = \lambda\dfrac{\partial u_0(x,t)}{\partial x} + \gamma u_0(x,t) \\ u_0(x,0) = \sin(\omega x) \end{cases} \\ \varepsilon^1: \begin{cases} \dfrac{\partial^\alpha u_1(x,t)}{\partial t^\alpha} = \dfrac{\partial^2 u_0(x,t)}{\partial x^2} + \lambda\dfrac{\partial u_1(x,t)}{\partial x} + \gamma u_1(x,t) \\ u_1(x,0) = 0 \end{cases} \\ \varepsilon^2: \begin{cases} \dfrac{\partial^\alpha u_2(x,t)}{\partial t^\alpha} = \dfrac{\partial^2 u_1(x,t)}{\partial x^2} + \lambda\dfrac{\partial u_2(x,t)}{\partial x} + \gamma u_2(x,t) \\ u_2(x,0) = 0 \end{cases} \\ \varepsilon^3: \begin{cases} \dfrac{\partial^\alpha u_3(x,t)}{\partial t^\alpha} = \dfrac{\partial^2 u_2(x,t)}{\partial x^2} + \lambda\dfrac{\partial u_3(x,t)}{\partial x} + \gamma u_3(x,t) \\ u_3(x,0) = 0 \end{cases} \\ \varepsilon^4: \begin{cases} \dfrac{\partial^\alpha u_4(x,t)}{\partial t^\alpha} = \dfrac{\partial^2 u_3(x,t)}{\partial x^2} + \lambda\dfrac{\partial u_4(x,t)}{\partial x} + \gamma u_4(x,t) \\ u_4(x,0) = 0 \end{cases} \\ ... \\ \varepsilon^n: \begin{cases} \dfrac{\partial^\alpha u_n(x,t)}{\partial t^\alpha} = \dfrac{\partial^2 u_{n-1}(x,t)}{\partial x^2} + \lambda\dfrac{\partial u_n(x,t)}{\partial x} + \gamma u_n(x,t); n \geq 1 \\ u_n(x,0) = 0; n \geq 1 \end{cases} \end{cases} \quad (2)$$

Let's solve the system ε^0:

$$\frac{\partial^\alpha u_0(x,t)}{\partial t^\alpha} = \lambda\frac{\partial u_0(x,t)}{\partial x} + \gamma u_0(x,t) \quad \text{with } u_0(x,0) = \sin(\omega x)$$

In Caputo's sense, the solution to this equation is in the form:

$$u_0(x,t) - u_0(x,0) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau, u_0(x,\tau))d\tau$$

As $f(\tau, u_0(x,\tau)) = \lambda\dfrac{\partial u_0(x,\tau)}{\partial x} + \gamma u_0(x,\tau)$,then,

$$u_0(x,t) - u_0(x,0) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(\lambda\frac{\partial u_0(x,\tau)}{\partial x} + \gamma u_0(x,\tau))d\tau$$

$$u_0(x,t) = \sum_{k=0}^{+\infty} u_{0_k}(x,t)$$

$$\text{Thus}: \begin{cases} u_{0_0}(x,t) = u_0(x,0) = \sin(\omega x) \\ u_{0_{n+1}}(x,t) = \dfrac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(\lambda\dfrac{\partial u_{0_n}(x,\tau)}{\partial x} + \gamma u_{0_n}(x,\tau))d\tau \end{cases}$$

We obtain after calculation:

$$\begin{cases} u_{0_0}(x,t) = \sin(\omega x) \\ u_{0_1}(x,t) = (\gamma\sin(\omega x) + \lambda\omega\cos(\omega x))\dfrac{t^\alpha}{\Gamma(\alpha+1)} \\ u_{0_2}(x,t) = (\gamma^2\sin(\omega x) - \lambda^2\omega^2\sin(\omega x) + 2\gamma\lambda\omega\cos(\omega x))\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ u_{0_3}(x,t) = (\gamma^3\sin(\omega x) - \lambda^3\omega^3\cos(\omega x) - 3\gamma\lambda^2\omega^2\sin(\omega x)) \\ +3\omega\gamma^2\lambda\cos(\omega x))\dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ u_{0_4}(x,t) = (\gamma^4\sin(\omega x) + \lambda^4\omega^4\sin(\omega x) - 4\lambda^3\gamma\omega^3\cos(\omega x) - 6\lambda^2\gamma^2\omega^2\sin(\omega x) \\ +4\lambda\gamma^3\omega\cos(\omega x))\dfrac{t^{4\alpha}}{\Gamma(4\alpha+1)} \\ ... \end{cases}$$

We have: $u_0(x,t) = \sum_{k=0}^{+\infty} u_{0_k}(x,t)$

Where from:

$$u_0(x,t) = \sin(\omega x)(1 + \frac{(\gamma t^\alpha)}{\Gamma(\alpha+1)} + \frac{(\gamma t^\alpha)^2}{\Gamma(2\alpha+1)} + \frac{(\gamma t^\alpha)^3}{\Gamma(3\alpha+1)} + \frac{(\gamma t^\alpha)^4}{\Gamma(4\alpha+1)} + ...)$$
$$+(\lambda\omega t^\alpha)\cos(\omega x))(\frac{1}{\Gamma(\alpha+1)} + 2\frac{(\gamma t^\alpha)}{\Gamma(2\alpha+1)} + 3\frac{(\gamma t^\alpha)^2}{\Gamma(3\alpha+1)} + 4\frac{(\gamma t^\alpha)^3}{\Gamma(4\alpha+1)} + ...)$$
$$-(\lambda\omega t^\alpha)^2\sin(\omega x)(\frac{1}{\Gamma(2\alpha+1)} + 3\frac{(\gamma t^\alpha)}{\Gamma(3\alpha+1)} + 6\frac{(\gamma t^\alpha)^2}{\Gamma(4\alpha+1)} + 10\frac{(\gamma t^\alpha)^3}{\Gamma(5\alpha+1)} + ...)$$
$$-(\lambda\omega t^\alpha)^3\cos(\omega x)((\frac{1}{\Gamma(3\alpha+1)} + 4\frac{(\gamma t^\alpha)}{\Gamma(4\alpha+1)} + 10\frac{(\gamma t^\alpha)^2}{\Gamma(5\alpha+1)} + ...)$$
$$+$$

...

We obtain:

$$u_0(x,t) = \sin(\omega x)\sum_{k=0}^{+\infty}\frac{(\gamma t^\alpha)^k}{\Gamma(k\alpha+1)} + (\lambda\omega t^\alpha)\cos(\omega x)\sum_{k=1}^{+\infty}\frac{k(\gamma t^\alpha)^{k-1}}{\Gamma(k\alpha+1)}$$
$$-(\lambda\omega t^\alpha)^2\sin(\omega x)\sum_{k=2}^{+\infty}\frac{k(k-1)(\gamma t^\alpha)^{k-2}}{2\Gamma(k\alpha+1)} - (\lambda\omega t^\alpha)^3\cos(\omega x)\sum_{k=3}^{+\infty}\frac{k(k-1)(k-2)(\gamma t^\alpha)^{k-3}}{6\Gamma(k\alpha+1)}$$
$$+...$$
$$\Longrightarrow u_0(x,t) = \sin(\omega x)E_\alpha(\gamma t^\alpha) + \cos(\omega x)\frac{(\lambda\omega t^\alpha)^1}{1!}E_\alpha^{(1)}(\gamma t^\alpha)$$
$$-\sin(\omega x)\frac{(\lambda\omega t^\alpha)^2}{2!}E_\alpha^{(2)}(\gamma t^\alpha) - \cos(\omega x)\frac{(\lambda\omega t^\alpha)^3}{3!}E_\alpha^{(3)}(\gamma t^\alpha) + ...$$

We finaly obtain:

$$\begin{cases} u_0(x,t) = \sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k\dfrac{(\lambda\omega t^\alpha)^{2k}}{(2k)!}E_\alpha^{(2k)}(\gamma t^\alpha) \\ \qquad + \cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k\dfrac{(\lambda\omega t^\alpha)^{2k+1}}{(2k+1)!}E_\alpha^{(2k+1)}(\gamma t^\alpha) \end{cases}$$

Let's solve the system ε^1 :

$$\frac{\partial^\alpha u_1(x,t)}{\partial t^\alpha} = \frac{\partial^2 u_0(x,t)}{\partial x^2} + \lambda\frac{\partial u_1(x,t)}{\partial x} + \gamma u_1(x,t) \quad \text{with } u_1(x,0) = 0$$

In Caputo's sense, the solution to this equation is in the form:

$$u_1(x,t) - u_1(x,0) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau, u_1(x,\tau))d\tau$$

As $f(\tau, u_1(x,\tau)) = \dfrac{\partial^2 u_0(x,t)}{\partial x^2} + \lambda\dfrac{\partial u_1(x,\tau)}{\partial x} + \gamma u_1(x,\tau)$,then,

$$u_1(x,t) - u_1(x,0) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(\frac{\partial^2 u_0(x,t)}{\partial x^2} + \lambda\frac{\partial u_1(x,\tau)}{\partial x} + \gamma u_1(x,\tau))d\tau$$

$$u_1(x,t) = \sum_{k=0}^{+\infty} u_{1_k}(x,t)$$

Thus:

$$\begin{cases} u_{1_0}(x,t) = u_1(x,0) = 0 \\ u_{1_{n+1}}(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(\frac{\partial^2 u_{0_n}(x,t)}{\partial x^2} + \lambda\frac{\partial u_{1_n}(x,\tau)}{\partial x} + \gamma u_{1_n}(x,\tau))d\tau \end{cases}$$

We obtain after calculation:

$$\begin{cases} u_{1_0}(x,t) = 0 \\ u_{1_1}(x,t) = -\omega^2 \sin(\omega x)\frac{t^\alpha}{\Gamma(\alpha+1)} \\ u_{1_2}(x,t) = (-2\lambda\omega^3\cos(\omega x) - 2\gamma\omega^2\sin(\omega x))\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ u_{1_3}(x,t) = (3\lambda^2\omega^4\sin(\omega x) - 3\gamma^2\omega^2\sin(\omega x) - 6\lambda\gamma\omega^3\cos(\omega x))\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ u_{1_4}(x,t) = (4\lambda^3\omega^5\cos(\omega x) - 4\gamma^3\omega^2\sin(\omega x) - 12\lambda\gamma^2\omega^3\cos(\omega x) \\ +12\lambda^2\gamma\omega^4\sin(\omega x))\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} \\ \ldots \end{cases}$$

We have: $u_1(x,t) = \sum_{k=0}^{+\infty} u_{1_k}(x,t)$

$$\begin{aligned} u_1(x,t) = &-(\omega^2 t^\alpha)\sin(\omega x)(\frac{1}{\Gamma(\alpha+1)} + \frac{2(\gamma t^\alpha)}{\Gamma(2\alpha+1)} + \frac{3(\gamma t^\alpha)^2}{\Gamma(3\alpha+1)} \\ &+\frac{4(\gamma t^\alpha)^3}{\Gamma(4\alpha+1)} + \frac{5(\gamma t^\alpha)^4}{\Gamma(5\alpha+1)} + \ldots) \\ &-(\lambda\omega^3 t^{2\alpha})\cos(\omega x))(\frac{2}{\Gamma(2\alpha+1)} + \frac{6(\gamma t^\alpha)}{\Gamma(3\alpha+1)} + \frac{12(\gamma t^\alpha)^2}{\Gamma(4\alpha+1)} + \frac{20(\gamma t^\alpha)^3}{\Gamma(5\alpha+1)} \\ &+\frac{30(\gamma t^\alpha)^4}{\Gamma(6\alpha+1)} + \ldots) \\ &+(\lambda^2\omega^4 t^{3\alpha})\sin(\omega x)(\frac{3}{\Gamma(3\alpha+1)} + 12\frac{(\gamma t^\alpha)}{\Gamma(4\alpha+1)} + 30\frac{(\gamma t^\alpha)^2}{\Gamma(5\alpha+1)} \\ &+60\frac{(\gamma t^\alpha)^3}{\Gamma(6\alpha+1)} + \ldots) \\ &+(\lambda^3\omega^5 t^{4\alpha})\cos(\omega x)(\frac{4}{\Gamma(4\alpha+1)} + 20\frac{(\gamma t^\alpha)}{\Gamma(5\alpha+1)} + 60\frac{(\gamma t^\alpha)^2}{\Gamma(6\alpha+1)} + \ldots) \\ &-(\lambda^4\omega^6 t^{5\alpha})\sin(\omega x)(\frac{5}{\Gamma(5\alpha+1)} + 30\frac{(\gamma t^\alpha)}{\Gamma(6\alpha+1)} + \ldots) \\ &+ \\ &\ldots \end{aligned}$$

We obtain:

$$\begin{aligned} u_1(x,t) = &-(\omega^2 t^\alpha)\sin(\omega x)\sum_{k=1}^{+\infty} k\frac{(\gamma t^\alpha)^{k-1}}{\Gamma(k\alpha+1)} \\ &-\frac{(\lambda\omega^3 t^{2\alpha})}{1}\cos(\omega x)\sum_{k=2}^{+\infty}\frac{k(k-1)(\gamma t^\alpha)^{k-2}}{\Gamma(k\alpha+1)} \\ &+\frac{(\lambda^2\omega^4 t^{3\alpha})}{2!}\sin(\omega x)\sum_{k=3}^{+\infty} k(k-1)(k-2)\frac{(\gamma t^\alpha)^{k-3}}{\Gamma(k\alpha+1)} \\ &+\frac{(\lambda^3\omega^5 t^{4\alpha})}{3!}\cos(\omega x)\sum_{k=4}^{+\infty} k(k-1)(k-2)(k-3)\frac{(\gamma t^\alpha)^{k-4}}{\Gamma(k\alpha+1)} \end{aligned}$$

$$-\frac{(\lambda^4\omega^6t^{5\alpha})}{4!}\sin(\omega x)\sum_{k=5}^{+\infty}k(k-1)(k-2)(k-3)(k-4)\frac{(\gamma t^\alpha)^{k-5}}{\Gamma(k\alpha+1)}$$
$$+$$

... This gives: $u_1(x,t) = -(\omega^2t^\alpha)[\sin(\omega x)E_\alpha^{(1)}(\gamma t^\alpha) + \frac{(\lambda\omega t^\alpha)}{1}\cos(\omega x)E_\alpha^{(2)}(\gamma t^\alpha)$

$$-\sin(\omega x)\frac{(\lambda\omega t^\alpha)^2}{2!}E_\alpha^{(3)}(\gamma t^\alpha) - \cos(\omega x)\frac{(\lambda\omega t^\alpha)^3}{3!}E_\alpha^{(4)}(\gamma t^\alpha) + \sin(\omega x)\frac{(\lambda\omega t^\alpha)^4}{4!}E_\alpha^{(5)}(\gamma t^\alpha) + ...]$$

We finaly obtain:

$$\begin{cases} u_1(x,t) = -(\omega^2t^\alpha)\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k\frac{(\lambda\omega t^\alpha)^{2k}}{(2k)!}E_\alpha^{(2k+1)}(\gamma t^\alpha) \\ \qquad -(\omega^2t^\alpha)\cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k\frac{(\lambda\omega t^\alpha)^{2k+1}}{(2k+1)!}E_\alpha^{(2k+2)}(\gamma t^\alpha) \end{cases}$$

Let's solve the system ε^2 :

$$\frac{\partial^\alpha u_2(x,t)}{\partial t^\alpha} = \frac{\partial^2 u_1(x,t)}{\partial x^2} + \lambda\frac{\partial u_2(x,t)}{\partial x} + \gamma u_2(x,t) \quad \text{with } u_2(x,0) = 0$$

Using the same approach of resolution as in ε^0 and $\varepsilon^1$and using the values of u_{1_k} obtained in ε^1, we obtain after calculation:

$$\begin{cases} u_{2_0}(x,t) = u_{2_1}(x,t) = 0 \\ u_{2_2}(x,t) = \omega^4\sin(\omega x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ u_{2_3}(x,t) = (3\lambda\omega^5\cos(\omega x) + 3\gamma\omega^4\sin(\omega x))\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ u_{2_4}(x,t) = (6\gamma^2\omega^4\sin(\omega x) - 6\lambda^2\omega^6\sin(\omega x) + 12\lambda\gamma\omega^5\cos(\omega x))\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} \\ u_{2_5}(x,t) = (10\gamma^3\omega^4\sin(\omega x) - 10\lambda^3\omega^7\cos(\omega x) + 30\lambda\gamma^2\omega^5\cos(\omega x) \\ -30\lambda^2\gamma\omega^6\sin(\omega x))\frac{t^{5\alpha}}{\Gamma(5\alpha+1)} \\ ... \end{cases}$$

With $u_2(x,t) = \sum_{k=0}^{+\infty}u_{2_k}(x,t)$, we obtain:

$$u_2(x,t) = (\omega^4t^{2\alpha})\sin(\omega x)(\frac{1}{\Gamma(2\alpha+1)} + \frac{3(\gamma t^\alpha)}{\Gamma(3\alpha+1)} + \frac{6(\gamma t^\alpha)^2}{\Gamma(4\alpha+1)} + \frac{10(\gamma t^\alpha)^3}{\Gamma(5\alpha+1)} + \frac{15(\gamma t^\alpha)^4}{\Gamma(6\alpha+1)} + ...)$$
$$+(\lambda\omega^5t^{3\alpha})\cos(\omega x))(\frac{3}{\Gamma(3\alpha+1)} + 12\frac{(\gamma t^\alpha)}{\Gamma(4\alpha+1)} + 30\frac{(\gamma t^\alpha)^2}{\Gamma(5\alpha+1)} + 60\frac{(\gamma t^\alpha)^3}{\Gamma(6\alpha+1)} + ...)$$
$$-(\lambda^2\omega^6t^{4\alpha})\sin(\omega x)(\frac{6}{\Gamma(4\alpha+1)} + 30\frac{(\gamma t^\alpha)}{\Gamma(5\alpha+1)} + 90\frac{(\gamma t^\alpha)^2}{\Gamma(6\alpha+1)} + ...)$$
$$-(\lambda^3\omega^7t^{5\alpha})\cos(\omega x)(\frac{10}{\Gamma(5\alpha+1)} + 60\frac{(\gamma t^\alpha)}{\Gamma(6\alpha+1)} + ...)$$
$$+$$

...

We have:

$$u_2(x,t) = \frac{(\omega^2t^\alpha)^2}{2}\sin(\omega x)\sum_{k=2}^{+\infty}\frac{k(k-1)(\gamma t^\alpha)^{k-2}}{\Gamma(k\alpha+1)}$$

$$+\frac{(\lambda\omega^5 t^{3\alpha})}{2}\cos(\omega x)\sum_{k=3}^{+\infty}k(k-1)(k-2)\frac{(\gamma t^\alpha)^{k-3}}{\Gamma(k\alpha+1)}$$
$$-\frac{(\lambda^2\omega^6 t^{4\alpha})}{2!2}\sin(\omega x)\sum_{k=4}^{+\infty}k(k-1)(k-2)(k-3)\frac{(\gamma t^\alpha)^{k-4}}{\Gamma(k\alpha+1)}$$
$$-\frac{(\lambda^3\omega^7 t^{5\alpha})}{3!2}\cos(\omega x)\sum_{k=5}^{+\infty}k(k-1)(k-2)(k-3)(k-4)\frac{(\gamma t^\alpha)^{k-5}}{\Gamma(k\alpha+1)}$$
$$+$$
$$...$$

This gives:

$$u_2(x,t)=\frac{(\omega^2 t^\alpha)^2}{2}[\sin(\omega x)E_\alpha^{(2)}(\gamma t^\alpha)+\frac{(\lambda\omega t^\alpha)}{1}\cos(\omega x)E_\alpha^{(3)}(\gamma t^\alpha)-\sin(\omega x)\frac{(\lambda\omega t^\alpha)^2}{2!}E_\alpha^{(4)}(\gamma t^\alpha)$$
$$-\cos(\omega x)\frac{(\lambda\omega t^\alpha)^3}{3!}E_\alpha^{(5)}(\gamma t^\alpha)+\sin(\omega x)\frac{(\lambda\omega t^\alpha)^4}{4!}E_\alpha^{(6)}(\gamma t^\alpha)+...]$$

We finaly obtain:

$$\begin{cases} u_2(x,t)=\dfrac{(\omega^2 t^\alpha)^2}{2}\sin(\omega x)\displaystyle\sum_{k=0}^{+\infty}(-1)^k\frac{(\lambda\omega t^\alpha)^{2k}}{(2k)!}E_\alpha^{(2k+2)}(\gamma t^\alpha) \\ \qquad +\dfrac{(\omega^2 t^\alpha)^2}{2}\cos(\omega x)\displaystyle\sum_{k=0}^{+\infty}(-1)^k\frac{(\lambda\omega t^\alpha)^{2k+1}}{(2k+1)!}E_\alpha^{(2k+3)}(\gamma t^\alpha) \end{cases}$$

Let's solve the system ε^3 :

$$\frac{\partial^\alpha u_3(x,t)}{\partial t^\alpha}=\frac{\partial^2 u_2(x,t)}{\partial x^2}+\lambda\frac{\partial u_3(x,t)}{\partial x}+\gamma u_3(x,t)\quad \text{with } u_3(x,0)=0$$

Using the same approach of resolution as in ε^0 and $\varepsilon^1$and using the values of u_{2_k} obtained in ε^2, we obtain after calculation:

$$\begin{cases} u_{3_0}(x,t)=u_{3_1}(x,t)=u_{3_2}(x,t)=0 \\ u_{3_3}(x,t)=-\omega^6\sin(\omega x)\dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ u_{3_4}(x,t)=(-4\lambda\omega^7\cos(\omega x)-4\gamma\omega^6\sin(\omega x))\dfrac{t^{4\alpha}}{\Gamma(4\alpha+1)} \\ u_{3_5}(x,t)=(10\lambda^2\omega^8\sin(\omega x)-10\gamma^2\omega^6\sin(\omega x)-20\lambda\gamma\omega^7\cos(\omega x))\dfrac{t^{5\alpha}}{\Gamma(5\alpha+1)} \\ u_{3_6}(x,t)=(20\lambda^3\omega^9\cos(\omega x)-20\gamma^3\omega^6\sin(\omega x)-60\lambda\gamma^2\omega^7\cos(\omega x) \\ +60\lambda^2\gamma\omega^8\sin(\omega x))\dfrac{t^{6\alpha}}{\Gamma(6\alpha+1)} \\ \quad ... \end{cases}$$

We have : $u_3(x,t)=\sum_{k=0}^{+\infty}u_{3_k}(x,t)$

$$u_3(x,t)=(-\omega^6 t^{3\alpha})\sin(\omega x)(\frac{1}{\Gamma(3\alpha+1)}+\frac{4(\gamma t^\alpha)}{\Gamma(4\alpha+1)}+\frac{10(\gamma t^\alpha)^2}{\Gamma(5\alpha+1)}$$
$$+\frac{20(\gamma t^\alpha)^3}{\Gamma(6\alpha+1)}+\frac{35(\gamma t^\alpha)^4}{\Gamma(7\alpha+1)}+...)$$
$$+(-\lambda\omega^7 t^{4\alpha})\cos(\omega x))(\frac{4}{\Gamma(4\alpha+1)}+20\frac{(\gamma t^\alpha)}{\Gamma(5\alpha+1)}+60\frac{(\gamma t^\alpha)^2}{\Gamma(6\alpha+1)}$$
$$+140\frac{(\gamma t^\alpha)^3}{\Gamma(7\alpha+1)}+...)$$
$$-(-\lambda^2\omega^8 t^{5\alpha})\sin(\omega x)(\frac{10}{\Gamma(5\alpha+1)}+60\frac{(\gamma t^\alpha)}{\Gamma(6\alpha+1)}+210\frac{(\gamma t^\alpha)^2}{\Gamma(7\alpha+1)}+...)$$
$$-(-\lambda^3\omega^9 t^{6\alpha})\cos(\omega x)(\frac{20}{\Gamma(6\alpha+1)}+140\frac{(\gamma t^\alpha)}{\Gamma(7\alpha+1)}+...)+...$$

We obtain:

$$u_3(x,t) = \frac{(-\omega^2 t^\alpha)^3}{3!}\sin(\omega x)\sum_{k=3}^{+\infty} k(k-1)(k-2)\frac{(\gamma t^\alpha)^{k-3}}{\Gamma(k\alpha+1)}$$
$$+\frac{(-\lambda\omega^7 t^{4\alpha})}{3!}\cos(\omega x)\sum_{k=4}^{+\infty} k(k-1)(k-2)(k-3)\frac{(\gamma t^\alpha)^{k-4}}{\Gamma(k\alpha+1)}$$
$$-\frac{(-\lambda^2\omega^8 t^{5\alpha})}{2!3!}\sin(\omega x)\sum_{k=5}^{+\infty} k(k-1)(k-2)(k-3)(k-4)\frac{(\gamma t^\alpha)^{k-5}}{\Gamma(k\alpha+1)}$$
$$-\frac{(-\lambda^3\omega^9 t^{6\alpha})}{3!3!}\cos(\omega x)\sum_{k=6}^{+\infty} k(k-1)(k-2)(k-3)(k-4)(k-5)\frac{(\gamma t^\alpha)^{k-6}}{\Gamma(k\alpha+1)}$$
$$+$$
$$...$$

This gives:

$$u_3(x,t) = \frac{(-\omega^2 t^\alpha)^3}{3!}[\sin(\omega x)E_\alpha^{(3)}(\gamma t^\alpha) + \frac{(\lambda\omega t^\alpha)}{1}\cos(\omega x)E_\alpha^{(4)}(\gamma t^\alpha) - \sin(\omega x)\frac{(\lambda\omega t^\alpha)^2}{2!}E_\alpha^{(5)}(\gamma t^\alpha)$$
$$-\cos(\omega x)\frac{(\lambda\omega t^\alpha)^3}{3!}E_\alpha^{(6)}(\gamma t^\alpha) + \sin(\omega x)\frac{(\lambda\omega t^\alpha)^4}{4!}E_\alpha^{(7)}(\gamma t^\alpha) + ...]$$

We finally obtain:

$$\begin{cases} u_3(x,t) = \dfrac{(-\omega^2 t^\alpha)^3}{3!}\sin(\omega x)\displaystyle\sum_{k=0}^{+\infty}(-1)^k\frac{(\lambda\omega t^\alpha)^{2k}}{(2k)!}E_\alpha^{(2k+3)}(\gamma t^\alpha) \\ \qquad + \dfrac{(-\omega^2 t^\alpha)^3}{3!}\cos(\omega x)\displaystyle\sum_{k=0}^{+\infty}(-1)^k\frac{(\lambda\omega t^\alpha)^{2k+1}}{(2k+1)!}E_\alpha^{(2k+4)}(\gamma t^\alpha) \end{cases}$$

Let's solve the system ε^4 :

$$\frac{\partial^\alpha u_4(x,t)}{\partial t^\alpha} = \frac{\partial^2 u_3(x,t)}{\partial x^2} + \lambda\frac{\partial u_4(x,t)}{\partial x} + \gamma u_4(x,t) \quad \text{with } u_4(x,0) = 0$$

Using the same approach of resolution as in ε^0 and $\varepsilon^1$and using the values of u_{3_k} obtained in ε^3, we obtain after calculation:

$$\begin{cases} u_{4_0}(x,t) = u_{4_1}(x,t) = u_{4_2}(x,t) = u_{4_3}(x,t) = 0 \\ u_{4_4}(x,t) = \omega^8\sin(\omega x)\dfrac{t^{4\alpha}}{\Gamma(4\alpha+1)} \\ u_{4_5}(x,t) = (5\lambda\omega^9\cos(\omega x) + 5\gamma\omega^8\sin(\omega x))\dfrac{t^{5\alpha}}{\Gamma(5\alpha+1)} \\ u_{4_6}(x,t) = (15\gamma^2\omega^8\sin(\omega x) - 15\lambda^2\omega^{10}\sin(\omega x) + 30\lambda\gamma\omega^9\cos(\omega x))\dfrac{t^{6\alpha}}{\Gamma(6\alpha+1)} \\ u_{4_7}(x,t) = (35\gamma^3\omega^8\sin(\omega x) - 35\lambda^3\omega^{11}\cos(\omega x) + 105\lambda\gamma^2\omega^9\cos(\omega x) \\ \qquad -105\lambda^2\gamma\omega^{10}\sin(\omega x))\dfrac{t^{7\alpha}}{\Gamma(7\alpha+1)} \\ \quad ... \end{cases}$$

We have: $u_4(x,t) = \sum_{k=0}^{+\infty} u_{4_k}(x,t)$

$$u_3(x,t) = (\omega^8 t^{4\alpha})\sin(\omega x)(\frac{1}{\Gamma(4\alpha+1)} + \frac{5(\gamma t^\alpha)}{\Gamma(5\alpha+1)} + \frac{15(\gamma t^\alpha)^2}{\Gamma(6\alpha+1)} + \frac{35(\gamma t^\alpha)^3}{\Gamma(7\alpha+1)} + \frac{70(\gamma t^\alpha)^4}{\Gamma(8\alpha+1)} + ...)$$
$$+(\lambda\omega^9 t^{5\alpha})\cos(\omega x))(\frac{5}{\Gamma(5\alpha+1)} + 30\frac{(\gamma t^\alpha)}{\Gamma(6\alpha+1)} + 105\frac{(\gamma t^\alpha)^2}{\Gamma(7\alpha+1)} + 280\frac{(\gamma t^\alpha)^3}{\Gamma(8\alpha+1)} + ...)$$
$$-(\lambda^2\omega^{10} t^{6\alpha})\sin(\omega x)(\frac{15}{\Gamma(6\alpha+1)} + 105\frac{(\gamma t^\alpha)}{\Gamma(7\alpha+1)} + 420\frac{(\gamma t^\alpha)^2}{\Gamma(8\alpha+1)} + ...)$$
$$-(\lambda^3\omega^{11} t^{7\alpha})\cos(\omega x)(\frac{35}{\Gamma(7\alpha+1)} + 280\frac{(\gamma t^\alpha)}{\Gamma(8\alpha+1)} + ...)$$

We obtain:

$$u_4(x,t) = \frac{(-\omega^2 t^\alpha)^4}{4!}\sin(\omega x)\sum_{k=4}^{+\infty} k(k-1)(k-2)(k-3)\frac{(\gamma t^\alpha)^{k-4}}{\Gamma(k\alpha+1)}$$
$$+\frac{(\lambda\omega^9 t^{5\alpha})}{4!}\cos(\omega x)\sum_{k=5}^{+\infty} k(k-1)(k-2)(k-3)(k-4)\frac{(\gamma t^\alpha)^{k-5}}{\Gamma(k\alpha+1)}$$
$$-\frac{(\lambda^2\omega^{10} t^{6\alpha}))}{2!4!}\sin(\omega x)\sum_{k=6}^{+\infty} k(k-1)(k-2)(k-3)(k-4)(k-5)\frac{(\gamma t^\alpha)^{k-6}}{\Gamma(k\alpha+1)}$$
$$-\frac{(\lambda^3\omega^{11} t^{7\alpha})}{3!4!}\cos(\omega x)\sum_{k=7}^{+\infty} k(k-1)(k-2)(k-3)(k-4)(k-5)(k-6)\frac{(\gamma t^\alpha)^{k-7}}{\Gamma(k\alpha+1)} + ...$$

We have:

$$u_4(x,t) = \frac{(-\omega^2 t^\alpha)^4}{4!}[\sin(\omega x)E_\alpha^{(4)}(\gamma t^\alpha) + \frac{(\lambda\omega t^\alpha)}{1}\cos(\omega x)E_\alpha^{(5)}(\gamma t^\alpha) - \sin(\omega x)\frac{(\lambda\omega t^\alpha)^2}{2!}E_\alpha^{(6)}(\gamma t^\alpha)$$
$$-\cos(\omega x)\frac{(\lambda\omega t^\alpha)^3}{3!}E_\alpha^{(7)}(\gamma t^\alpha) + \sin(\omega x)\frac{(\lambda\omega t^\alpha)^4}{4!}E_\alpha^{(8)}(\gamma t^\alpha) + ...]$$

We finally obtain:

$$\begin{cases} u_4(x,t) = \dfrac{(-\omega^2 t^\alpha)^4}{4!}\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k\dfrac{(\lambda\omega t^\alpha)^{2k}}{(2k)!}E_\alpha^{(2k+4)}(\gamma t^\alpha) \\ \qquad + \dfrac{(-\omega^2 t^\alpha)^4}{4!}\cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k\dfrac{(\lambda\omega t^\alpha)^{2k+1}}{(2k+1)!}E_\alpha^{(2k+5)}(\gamma t^\alpha) \end{cases}$$

From close to close, we obtain:

$$\begin{cases} u_p(x,t) = \dfrac{(-\omega^2 t^\alpha)^p}{p!}\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k\dfrac{(\lambda\omega t^\alpha)^{2k}}{(2k)!}E_\alpha^{(2k+p)}(\gamma t^\alpha) \\ \qquad + \dfrac{(-\omega^2 t^\alpha)^p}{p!}\cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k\dfrac{(\lambda\omega t^\alpha)^{2k+1}}{(2k+1)!}E_\alpha^{(2k+p+1)}(\gamma t^\alpha); p \geq 0 \end{cases} \quad (3)$$

Let's posing: $\varphi_n(x,t) = \sum_{p=0}^{n} \varepsilon^p u_p(x,t)$, with: $u(x,t) = \lim_{n\to+\infty} \varphi_n(x,t)$

The solution of the equation (P_3) is:

$$u(x,t) = \lim_{n\to+\infty}\sum_{p=0}^{n}\varepsilon^p\ [\frac{(-\omega^2 t^\alpha)^p}{p!}\sin(\omega x)\sum_{k=0}^{+\infty}(-1)^k\frac{(\lambda\omega t^\alpha)^{2k}}{(2k)!}E_\alpha^{(2k+p)}(\gamma t^\alpha)$$
$$+\frac{(-\omega^2 t^\alpha)^p}{p!}\cos(\omega x)\sum_{k=0}^{+\infty}(-1)^k\frac{(\lambda\omega t^\alpha)^{2k+1}}{(2k+1)!}E_\alpha^{(2k+p+1)}(\gamma t^\alpha)\ ]$$
$$\Longrightarrow u(x,t) = \sin(\omega x)\lim_{n\to+\infty}\sum_{p=0}^{n}\left(\frac{(-\varepsilon\omega^2 t^\alpha)^p}{p!}\sum_{k=0}^{+\infty}(-1)^k\frac{(\lambda\omega t^\alpha)^{2k}}{(2k)!}E_\alpha^{(2k+p)}(\gamma t^\alpha)\right)$$
$$+\cos(\omega x)\lim_{n\to+\infty}\sum_{p=0}^{n}\left(\frac{(-\varepsilon\omega^2 t^\alpha)^p}{p!}\sum_{k=0}^{+\infty}(-1)^k\frac{(\lambda\omega t^\alpha)^{2k+1}}{(2k+1)!}E_\alpha^{(2k+p+1)}(\gamma t^\alpha)\right)$$

We finaly obtain:

$$\begin{cases} u(x,t) = \sin(\omega x)\lim_{n\to+\infty}\sum_{p=0}^{n}\left(\dfrac{(-\varepsilon\omega^2 t^\alpha)^p}{p!}\sum_{k=0}^{+\infty}(-1)^k\dfrac{(\lambda\omega t^\alpha)^{2k}}{(2k)!}E_\alpha^{(2k+p)}(\gamma t^\alpha)\right) \\ \qquad + \cos(\omega x)\lim_{n\to+\infty}\sum_{p=0}^{n}\left(\dfrac{(-\varepsilon\omega^2 t^\alpha)^p}{p!}\sum_{k=0}^{+\infty}(-1)^k\dfrac{(\lambda\omega t^\alpha)^{2k+1}}{(2k+1)!}E_\alpha^{(2k+p+1)}(\gamma t^\alpha)\right) \end{cases}$$

Note: For $\alpha = 1$, we have: $E_\alpha^{(2k+p)}(\gamma t^\alpha) = E_\alpha^{(2k+p)}(\gamma t^\alpha) = e^{\gamma t}$

$$u(x,t) = e^{(\gamma-\varepsilon\omega^2)t}\sin(\omega x + \lambda\omega t) \quad (4)$$

Thus, this solution is the same as that obtain by Y. Minougou in his thesis [1], [2], [3], [4], [5] where he used the Adomian method with $\alpha = 1$.

3.2 Numerical results

We take $\alpha = 1$

$If \quad \omega = 1$; $\varepsilon = 0.1$; $\gamma = 1$ and $\lambda = 1$, the approximate values of $u(x,t)$, with x in line and t in column, are:

x \ t	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**
0.1	0.04860	0.14460	0.28581	0.46916	0.69069
0.2	0.07230	0.19054	0.35187	0.55255	0.78802
0.3	0.09527	0.23458	0.41441	0.63042	0.87749
0.4	0.11729	0.27627	0.47281	0.70199	0.95812
0.5	0.13814	0.31521	0.52650	0.76655	1.02931

Thus, we find that $u(x,t)$ is an increasing function of x and t.

4 Conclusion

The resolution of fractional equations by the Perturbation method regular is very tedious because it requires much more vigilance in calculations. The use of the Mittag-Leffler function is essential. In the event that $\alpha = 1$, the solutions become simpler because they appeal to the exponential function. In short, the method, although tedious, is very effective because it provides solutions to most linear partial differential equations. However, it requires mastery of the basic notions of numerical series. In addition, we have in perspective to apply it to the nonlinear equations.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Kanth AR, Kumar PM. Computational results and analysis for a class of linear and nonlinear singularly perturbed convection delay problems on Shishkin mesh. Hacettepe Journal of Mathematics and Statistics. 2020;49(1):221-35.

[2] Kumar P, Ravi Kanth AS. Computational study for a class of time-dependent singularly perturbed parabolic partial differential equation through tension spline. Computational and Applied Mathematics. 2020;39(3):1-9.

[3] Palli MM, Ravi Kanth AS. Numerical simulation for a class of singularly perturbed convection delay problems. Khayyam Journal of Mathematics. 2021;7(1):52-64.

[4] Kanth ASVR, Kumar PMM. A Numerical Algorithm Based on Tension Spline Scheme for Solving Singularly Perturbed Differential Difference Equations with Shifts; 2022.

[5] Minougou Y. (Contribution des méthodes SBA et Laplace-SBA à la résolution des modèles de convection-diffusion-réaction : Thèse de doctorat Université Joseph KI-ZERBO, soutenu le 10-06-2019; 2019.

[6] Seny O, Abbo B, Yaro R, PARE Y. Convergence of new approch of Adomian's method for solving volterra fractional integral of second kind , UFR/SEA, Departement Mathematiques et Informatique (Burkina Faso); 2020.

[7] Bationo YJ, F. Bassono. The solutions of the linear fractional diffusion and diffusion-convection equations via the Regular Perturbation Method (RPM), University Joseph Ki Zerbo; 2022.

[8] Youssouf P, Bassono F, Some B. A new technique for numerical resolution of few nonlinear integral equations of fredholm by sba method. International Journal of Applied Mathematical Research. 2012;70(1):21-33.

[9] Mbaiguesse D., Jeremie B, Abbo B, Pare Y. UniversityJoseph KI-ZERBO; University of Njamena Tchad**Analytical solution of some nonlinear fractional integrodifferential equations of the Fedholm second kind by a new approximation technique of the numerical SBA method, KAMATE Adama; 2022.

[10] Ali K. Résolution des é quations aux dérivées partielles linéaires et non-linéaires moyennant des approches analytiques.Extension aux cas d'EDP d'ordre fractionnaire.Thèse de doctorat en science, mathématiques appliqu ée, UNIVERSITE Ferhat ABBAS SETIF 1; 2019.

[11] Bassono F. Résolution numérique compar ée de quelques équations fonctionnelles par la méthode SBA, d écompositionnelle d'Adomia et Perturbations, thèse de doctorat unique, Université de Ouagadougou, janvier 2013, UFR SEA; 2013.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
https://www.sdiarticle5.com/review-history/93804