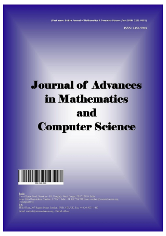

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The α-analogues of r-Whitney Numbers via Normal Ordering

Hye Kyung Kim [a*]

[a] *Department of Mathematics Education, Daegu Catholic University, Gyeongsan 38430, Republic of Korea.*

Authors' contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

The normal ordering of an integral power of the number operator $a^\dagger a$ in terms of boson annihilation a and creation $a^\dagger$ operators is expressed with the help of the Stirling numbers of the second kind. The normal ordering problems directly links the problems to combinatorics. Whit this in mind, in this paper, we define the α-analogues of r-Whitney numbers of the first kind and those of second kind, which are different from degenerate r-Whitney numbers. We show that α-analogues falling factorial of the number operator is expressed in terms of the α-analogues of r-Whitney numbers of the first kind and its inverse formula is expressed as those of the second kind. We also derived some properties, recurrence relations and several identities on those numbers arising from Boson annihilation a and creation operators $a^\dagger$, number operators $\hat{h}$ and coherent states.

Keywords: α-analogues of r-Stirling numbers of the first kind; α-analogues of r-Stirling numbers of the second kind; r-Whitney numbers of the first kind; r-Whitney numbers of the second kind; Normal ordering.

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1 Introduction

Dowling constructed Dowling lattice $Q_n(G)$ [1] for a finite group of order m using the Möbious function. He also introduced the Whitney numbers of the first and second kind, $\omega_m(n,k)$ and $W_m(n,k)$ $\quad (n \geq k \geq 0, m \geq 1)$

Corresponding author: E-mail: hkkim@cu.ac.kr;

respectively which are independent if the group G itself, but depend only on its order m. Benoumhani [2, 3] studied some properties of the Whitney numbers of both kind. Mezö [4] introduced the r-Whitey numbers of the first and second kind, respectively as follows.

For $r \in \mathbb{N}$, the r-Whitney numbers of the first kind $\omega_m^{(r)}(n,k)$ and those of the second kind $W_m^{(r)}(n,k)$ are given by

$$m^n(x)_n = \sum_{k=0}^{n}(-1)^{n-k}\omega_m^{(r)}(n,k)(mx+r)^k, \qquad \text{(see [1, 3-5])} \tag{1.1}$$

and

$$(mx+r)^n = \sum_{k=0}^{n} m^k W_m^{(r)}(n,k)(x)_k, \qquad (n \geq 0), \qquad \text{(see [1, 3-5])}. \tag{1.2}$$

Many scholars have derived interesting results about r-Whitney numbers [5-9]. Belbachir and Bousbaa [10] introduced the translated Whitney numbers of the second kind.

Recently, some mathematicians have studied the counting sequences problems derived from Boson operators including the λ-analogues of r-Stirling numbers of the second kind, the analogue of r-Stirling numbers, the degenerate Stirling numbers, the degenerate r-Bell polynomials, the degenerate r-Whitney numbers and degenerate r-Dowling polynomials (see [8, 11-19]).

The novelty of this paper is that the α-analogues of r-Whitney numbers of the first kind and those of second kind are introduced which are the r-Whitney numbers of the first and second kind, respectively when $\alpha \to 1$. These new numbers are different from the translated r-Whitney numbers [10] and the degenerate r-Whitney numbers [8]. We derived some properties, recurrence relations and several identities on those numbers arising by Boson annihilation a and creation operators $a^\dagger$, normal ordering $a^\dagger a$, number operators $\hat{h}$ and coherent states. Here, we further mention the novelty of this paper by comparing the results of the work [8] written in same spirit as the present work.

From one of results in this manuscript, we show that an integral power of the number operators is represented by means of the α-analogues of r-Whitney numbers of the second kind $W_{m,\lambda}^{(r)}(k,j)$ and the α-analogues integral power of the number operators (see Theorem 3 (ii)) as follows:

$$(m\hat{h}+r)^k = \sum_{j=0}^{k} W_{m,\alpha}^{(r)}(k,j)m^j(\hat{h})_{j,\alpha}. \tag{1.3}$$

From (1.3), when $\alpha = 1$, we have

$$(m\hat{h}+r)^k = \sum_{j=0}^{k} W_m^{(r)}(k,j)m^j(\hat{h})_j. \tag{1.4}$$

However, in [8], it noted that a degenerate integral power of the number operators is represented by means of the degenerate r-Whitney numbers of the second kind $W_{m,\lambda}^{(r)}(k,j)$ and the normal ordering $(a^\dagger)^k a^k$ as follows:

$$(m\hat{h}+r)_{k,\lambda} = \sum_{j=0}^{k} W_{m,\lambda}^{(r)}(k,j)m^j(a^\dagger)^k a^k. \tag{1.5}$$

From (1.5), when $\lambda = 0$, we get

$$(m\hat{h}+r)^k = \sum_{j=0}^{k} W_m^{(r)}(k,j)m^j(a^\dagger)^k a^k. \tag{1.6}$$

Comparing (1.4) and (1.6), we obtain two different identities. We can also observe that several different representations are obtained by comparing the identities when $\alpha = 1$ in this manuscript with the identities when $\lambda = 0$ in [8] for the degenerate r-Whitney numbers.

The outline of this paper is as follows. In Section 1, we recall the α-analogues of r-Stirling numbers of the first and second kind and the generating functions of them. We remind the reader of the Boson operators, the number operators, the normal ordering of an integral power of the number operator in terms of Boson operators and the α-analogues of integral power of the number operator in terms of Boson operators. Section 2 is the main result of this paper. We state that the α-analogues integral power of number operator $(\hat{n})_{n,\alpha}$ and $(m\hat{n}+r)^n$ in terms of the number operators with the coefficients the α-analogues of r-Whitney numbers of the first kind in Theorem 2.1 (i) and its inversion formula in Theorem 2.1 (ii) with the coefficients the α-analogues of r-Whitney numbers of the second kind.

We also derive some properties, recurrence relation and several combinatorial identities including the α-analogues of r-Whitney numbers of both kinds from identities in Theorem 2.1, number operators $\hat{h}$ and coherent states, etc.

We first introduce the definitions and properties needed this manuscript.

For $\alpha \in \mathbb{R}$, the generalized falling factorial sequence is given by

$$(x)_{n,\alpha} = x(x-\alpha)\cdots(x-(n-1)\alpha) \quad \text{and} \quad (x)_{0,\alpha} = 1, \qquad (n \geq 1). \tag{1.7}$$

For convenience, when $\alpha \to 1$, $(x)_{n,\alpha}$ are called the α-analogues integral power of x. When $\alpha \to 0$, $(x)_{n,\alpha}$ are called the degenerate integral power of x. We note that $\lim_{\alpha\to 1}(x)_{n,\alpha} = (x)_n$ and $\lim_{\alpha\to 0}(x)_{n,\alpha} = x^n$.

The Stirling numbers of the first kind are given by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l \quad \text{and} \quad \frac{1}{k!}\big(\log(1+t)\big)^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \qquad \text{(see [12, 20-23])}. \tag{1.8}$$

For $r \in \mathbb{N}\cup\{0\}$, the α-analogues of r-Stirling numbers of the first kind are defined by

$$(x+r)_{n,\alpha} = \sum_{k=0}^{n} S_{1,\alpha}^{(r)}(n+r,k+r)x^k, \qquad (n \geq 0), \qquad \text{(see [22, 24])}. \tag{1.9}$$

The generating function of the α-analogues of r-Stirling numbers of the first kind is

$$(1+\alpha t)^{\frac{r}{\alpha}}\frac{1}{k!}\left(\frac{\log(1+\alpha t)}{\alpha}\right)^k = \sum_{n=k}^{\infty} S_{1,\alpha}^{(r)}(n+r,k+r)\frac{t^n}{n!}, \qquad \text{(see [22, 24])}. \tag{1.10}$$

When $\alpha \to 1$, we have the r-Stirling numbers of the first kind as follows:

$$(x+r)_n = \sum_{k=0}^{n} S_1^{(r)}(n+r,k+r)x^k, \qquad (n \geq 0), \qquad \text{(see [19, 21-24])}.$$

When $r = 0$, we have the α-analogues of the Stirling numbers of the first kind as follows:

$$(x)_{n,\alpha} = \sum_{k=0}^{n} S_{1,\alpha}(n,k)x^k \quad \text{and} \quad \frac{1}{k!}\left(\frac{\log(1+\alpha t)}{\alpha}\right)^k = \sum_{n=k}^{\infty} S_{1,\alpha}(n,k)\frac{t^n}{n!}, \quad \text{(see [22, 24])}. \tag{1.11}$$

The Stirling numbers of the second kind $S_2(n,k)$ are given by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l \quad \text{and} \quad \frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \qquad \text{(see [12, 13, 21, 25-27])}. \tag{1.12}$$

The Bell polynomials are given by

$$bel_n(x) = \sum_{k=0}^{n} S_2(n,k)x^k \quad \text{and} \quad \sum_{n=0}^{\infty} bel_n(x)\frac{t^n}{n!} = e^{x(e^t-1)} \qquad \text{(see [12, 21, 26])}. \tag{1.13}$$

The r-Stirling numbers of the second kind $S_2^{(r)}(n,k)$ are given by

$$e^{rt}\frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2^{(r)}(n,k)\frac{t^n}{n!}, \qquad \text{(see [12, 19, 21, 23, 25, 26])}. \tag{1.14}$$

The r-Bell polynomials are given by

$$\sum_{n=0}^{\infty} bel_n^{(r)}(x)\frac{t^n}{n!} = e^{x(e^t-1)+rt} \qquad \text{(see [12, 19, 21, 26])}. \tag{1.15}$$

As the inversion formula of (1.9), the α-analogues of r-Stirling numbers of the second kind are given by

$$(x+r)^n = \sum_{k=0}^{n} S_{2,\alpha}^{(r)}(n+r,k+r)(x)_{k,\alpha}, \qquad (n \geq 0), \qquad \text{(see [25])} \tag{1.16}$$

and the generating function

$$\frac{1}{k!}\left(\frac{e^{\alpha t}-1}{\alpha}\right)^k e^{rt} = \sum_{n=k}^{\infty} S_{2,\alpha}^{(r)}(n+r,k+r)\frac{t^n}{n!}, \qquad \text{(see [25])}. \tag{1.17}$$

When $r = 0$, as the inversion formula of (1.11), the α-analogues of Stirling numbers of the second kind are given by

$$x^n = \sum_{k=0}^{n} S_{2,\alpha}(n,k)(x)_{k,\alpha}, \tag{1.18}$$

and the generating function

$$\frac{1}{k!}\left(\frac{e^{\alpha t}-1}{\alpha}\right)^k = \sum_{n=k}^{\infty} S_{2,\alpha}(n,k)\frac{t^n}{n!}, \quad (n \geq 0), \quad \text{(see [25])}.$$

We have the r-Stirling numbers of the second kind as follows:

$$(x+r)^n = \sum_{k=0}^{n} S_2^{(r)}(n+r,k+r)(x)_k, \qquad (n \geq 0), \qquad \text{(see [19, 23, 25])}.$$

Let a and $a^\dagger$ be the boson annihilation and creation operators satisfying the commutation relation

$$[a,a^\dagger] = aa^\dagger - a^\dagger a = 1, \quad \text{(see [11, 13-15])}. \tag{1.19}$$

The number states $|l\rangle,\ l = 1,2,\ldots$, are defined as

$$a|l\rangle = \sqrt{l}|l-1\rangle, \quad a^\dagger|l\rangle = \sqrt{l+1}|l+1\rangle, \qquad \text{(see [2, 5, 13, 24])}. \tag{1.20}$$

The number operator is defined by

$$\hat{h}|l\rangle = l|l\rangle, \quad (k \geq 0), \quad \text{(see [2, 5, 13])}. \tag{1.21}$$

By (1.20) and (1.21), we get $\hat{h} = a^\dagger a$.

The coherent states $|z\rangle$, where z is a complex number, satisfy $a|z\rangle = z|z\rangle$, $z|z\rangle = 1$. To show a connection to coherent states, we recall that the harmonic oscillator has Hamiltonian $\hat{n} = a^\dagger a$ (neglecting the zero point energy) and the usual eigenstates $|n\rangle$ (for $n \in \mathbb{N}$) satisfying $\hat{n}|n\rangle = n|n\rangle$ and $\langle l|n\rangle = \delta_{l,n}$, where $\delta_{l,n}$ is the Kronecker's symbol.

The evaluation of the expectation values of both sides of (1.20) with respect to the coherent state $|z\rangle$ yields

$$\langle z|(a^\dagger a)^k|z\rangle = \sum_{l=0}^{k} S_2(k,l)|z|^{2l}, \qquad \text{(see [13-15, 24])}. \tag{1.22}$$

For any polynomials $f(x)$, we have

$$\frac{d}{dx}(x + f(x)) = f(x) + x\frac{d}{dx}f(x). \tag{1.23}$$

From (1.23), we get $[\frac{d}{dx},\ x] = [a,\ a^\dagger] = 1$ and use both forms by identifying formally

$a = \frac{d}{dx}$ and $a^\dagger = x$.

When we consider the action of $(x\frac{d}{dx})^n$ on a polynomials $f(x)$, we have

$$\left(x\frac{d}{dx}\right)^n f(x) = \sum_{k=0}^{n} S_2(n,k)x^k\left(\frac{d}{dx}\right)^k f(x), \qquad \text{(see [11-14, 17, 18, 24])}, \tag{1.24}$$

or, alternatively

$$(a^\dagger a)^n = \sum_{k=0}^{n} S_2(n,k)(a^\dagger)^k a^k, \qquad \text{(see [11-14, 17, 18, 24])}. \tag{1.25}$$

The α-analogues of differential operator is given by

$$\left(x\frac{d}{dx}\right)_{k,\alpha} = \left(x\frac{x}{dx}\right)\left(x\frac{d}{dx} - \alpha\right)\cdots\left(x\frac{d}{dx} - (k-1)\alpha\right) \quad \text{(see [28])}. \tag{1.26}$$

From (1.26), we have

$$\left(x\frac{d}{dx}\right)_{k,\alpha} x^n = (n)_{k,\alpha}x^n, \quad \text{(see [28])}. \tag{1.27}$$

When $\alpha \to 1$, we note that $\left(x\frac{d}{dx}\right)_k x^n = (n)_k x^n$.
Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. In view of (1.24), by (1.9) and (1.27), we observe that

$$\begin{aligned}(\hat{h}+1)_{k,\alpha} = (a^\dagger a + r)_{k,\alpha} &= \sum_{j=0}^{k} S_{1,\alpha}^{(r)}(k+r, j+r)(a^\dagger a)^j \\ &= \sum_{j=0}^{k} S_{1,\alpha}^{(r)}(k+r, j+r)(\hat{h})^j, \quad \text{(see [24])}.\end{aligned} \tag{1.28}$$

When $r = 0$, we have

$$(\hat{h})_{k,\alpha} = (a^\dagger a)_{k,\alpha} = \sum_{j=0}^{k} S_{1,\alpha}(k,j)(a^\dagger a)^j = \sum_{j=0}^{k} S_{1,\alpha}(k,j)(\hat{h})^j, \quad \text{(see [24])}. \tag{1.29}$$

When $\alpha \to 1$, we obtain an identity different from the identity obtain in [18] as follows:

$$(\hat{h})_k = \sum_{j=0}^{k} S_1(k,j)(\hat{h})^j. \tag{1.30}$$

To obtain the inverse formula of (1.16), from (1.27), we observe that

$$\begin{aligned}(\hat{h})^k = (a^\dagger a + r)^k &= \sum_{j=0}^{k} S_{2,\alpha}^{(r)}(k+r, j+r)(a^\dagger a)_{j,\alpha} \\ &= \sum_{j=0}^{k} S_{2,\alpha}^{(r)}(k+r, j+r)(\hat{h})_{j,\alpha}, \quad \text{(see [24])}.\end{aligned} \tag{1.31}$$

Form (1.29) and (1.31), we get

$$(\hat{h}+r)^k = \sum_{j=0}^{k}\sum_{l=0}^{j} S_{2,\alpha}^{(r)}(k+r, j+r) S_{1,\alpha}(j,l)(\hat{h})^l, \quad \text{(see [24])}. \tag{1.32}$$

When $r = 0$, from (1.16) and (1.27), we have the inverse formula of (1.29) as follows:

$$(\hat{h})^k = \sum_{j=0}^{k} S_{2,\alpha}(k,j)(\hat{h})_{j,\alpha}, \quad \text{(see [24])}. \tag{1.33}$$

When $\alpha \to 1$, we obtain an identity different from the identity obtain in [18] as follows:

$$(\hat{h})^k = \sum_{j=0}^{k} S_2(k,j)(\hat{h})_j. \tag{1.34}$$

2 α-analogues of r-Whitney Numbers of the First and Second Kind via Normal Ordering

In view of (1.1) and (1.2), we define by the α-analogues of r-Whitney numbers of the first kind $\omega_{m,\alpha}^{(r)}(n,k)$ and those $W_{m,\alpha}^{(r)}(n,k)$, respectively.

$$m^n (x)_{n,\alpha} = \sum_{k=0}^{n} (-1)^{n-k} \omega_{m,\alpha}^{(r)}(n,k)(mx+r)^k \tag{2.1}$$

and

$$(mx+r)^n = \sum_{k=0}^{n} W_{m,\alpha}^{(r)}(n,k) m^k (x)_{k,\alpha}, \qquad (n \ \geq \ 0). \tag{2.2}$$

Note that

$$\lim_{\alpha\to 1} \omega_{m,\alpha}^{(r)}(n,k) = \omega_m^{(r)}(n,k) \quad \text{and} \quad \lim_{\alpha \to 1} W_{m,\alpha}^{(r)}(n,k) = W_m^{(r)}(n,k).$$

Proposition 2.1. *[25] For $n, k \in \mathbb{N} \cup \{0\}$ with $n \geq k$, we have the generating function of the α-analogues of r-Whitney of the second kind as*

$$\sum_{n=k}^{\infty} W_{m,\alpha}^{(r)}(n,k)\frac{t^n}{n!} = e^{rt}\frac{1}{k!}\frac{1}{\alpha^k}\left(\frac{e^{\alpha mt}-1}{m}\right)^k.$$

Proposition 2.2. *For $n, k \in \mathbb{N} \cup \{0\}$ with $n \geq k$, we have the generating function of the α-analogues of r-Whitney of the first kind as*

$$e_{\alpha m}^{-r}(t)\frac{1}{k!}\frac{1}{\alpha^k}\left(\frac{\log(1+\alpha mt)}{m}\right)^k = \sum_{n=k}^{\infty}(-1)^{n-k}\omega_{m,\alpha}^{(r)}(n,k)\frac{t^n}{n!},$$

where $e_{\alpha m}^{-r}(t) = (1+\alpha mt)^{-\frac{r}{\alpha m}}$.

Proof. From (2.1), we observe that

$$\begin{aligned}
\sum_{k=0}^{\infty}\left\{\sum_{n=k}^{\infty}(-1)^{n-k}\omega_{m,\alpha}^{(r)}(n,k)\frac{t^n}{n!}\right\}(mx+r)^k &= \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}(-1)^{n-k}\omega_{m,\alpha}^{(r)}(n,k)(mx+r)^k\right\}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} m^n (x)_{n,\alpha}\frac{t^n}{n!} = (1+\alpha mt)^{\frac{x}{\alpha}} \\
&= (1+\alpha mt)^{\frac{mx+r}{m\alpha}-\frac{r}{m\alpha}} \\
&= (1+\alpha mt)^{-\frac{r}{\alpha m}}\exp\left(\frac{mx+r}{m\alpha}\log(1+\alpha mt)\right) \\
&= e_{\alpha m}^{-r}(t)\sum_{k=0}^{\infty}\left(\frac{mx+r}{m\alpha}\right)^k\big(\log(1+\alpha mt)\big)^k\frac{1}{k!} \\
&= e_{\alpha m}^{-r}(t)\sum_{k=0}^{\infty}\frac{1}{k!}\frac{1}{\alpha^k}\left(\frac{\log(1+\alpha mt)}{m}\right)^k(mx+r)^k.
\end{aligned} \tag{2.3}$$

By comparing of the coefficients of both sides of (2.3), we have the desire the generating function.
■

When $m = 1$, for $r \geq 1$, by (1.11), (1.18), Proposition 1 and Proposition 2, we note that

$$(-1)^{n-k}\omega_{1,\alpha}^{(r)}(n,k) = S_{1,\alpha}^{(-r)}(n+r,k+r) \quad \text{and} \quad W_{1,\alpha}^{(r)}(n,k) = S_{2,\alpha}^{(r)}(n+r,k+r).$$

Theorem 2.1. *For $n, j \in \mathbb{N} \cup \{0\}$ with $n \geq j$, we have*

$$(i) \quad m^n(\hat{h})_{n,\alpha} = \sum_{j=0}^{n}(-1)^{n-j}\omega_{m,\alpha}^{(r)}(n,j)(m\hat{h}+r)^j.$$

The inverse formula of (i) is

$$(ii) \quad (m\hat{h}+r)^n = \sum_{j=0}^{n} W_{m,\alpha}^{(r)}(n,j)m^j(\hat{h})_{j,\alpha}.$$

Proof. From (2.1) and (1.27), we note that

$$\begin{aligned} m^k\left(x\frac{d}{dx}\right)_{k,\alpha}f(x) &= \sum_{n=0}^{\infty}a_n m^k\left(x\frac{d}{dx}\right)_{k,\alpha}x^n = \sum_{n=0}^{\infty}a_n m^k(n)_{k,\alpha}x^n \\ &= \sum_{n=0}^{\infty}a_n\sum_{j=0}^{k}(-1)^{k-j}\omega_{m,\alpha}^{(r)}(k,j)(mn+r)^j x^n \\ &= \sum_{j=0}^{k}(-1)^{k-j}\omega_{m,\alpha}^{(r)}(k,j)\left(mx\frac{d}{dx}+r\right)^j f(x). \end{aligned} \tag{2.4}$$

Thus, by (2.4), we have

$$m^k(a^\dagger a)_{k,\alpha} = \sum_{j=0}^{k}(-1)^{k-j}\omega_{m,\alpha}^{(r)}(k,j)(ma^\dagger a+r)^j. \tag{2.5}$$

To obtain the inverse formula of (2.5), from (2.2) and (1.27), we observe that

$$\begin{aligned} \left(mx\frac{d}{dx}+r\right)^k f(x) &= \left(mx\frac{d}{dx}+r\right)^k\sum_{n=0}^{\infty}a_n x^n = \sum_{n=0}^{\infty}a_n(mn+r)^k x^n \\ &= \sum_{n=0}^{\infty}a_n\sum_{j=0}^{k}W_{m,\alpha}^{(r)}(k,j)m^j(n)_{j,\alpha}x^n \\ &= \sum_{j=0}^{k}W_{m,\alpha}^{(r)}(k,j)m^j\left(x\frac{d}{dx}\right)_{j,\alpha}f(x). \end{aligned} \tag{2.6}$$

By (2.6), we get

$$(ma^\dagger a+r)^k = \sum_{j=0}^{k}W_{m,\alpha}^{(r)}(k,j)m^j(a^\dagger a)_{j,\alpha}. \tag{2.7}$$

By (2.5) and (2.7), we have the desired result.

■

Theorem 2.2. *For $n, j \in \mathbb{N}\cup\{0\}$ with $n \geq j$, we have*

$$\omega_{m,\alpha}^{(0)}(n,j) = m^{n-j}S_{1,\alpha}(n,j) \quad \text{and} \quad W_{m,\alpha}^{(0)}(n,j) = m^{n-j}S_{2,\alpha}(n,j).$$

Proof. From (1.29) and Theorem 2.1 (i), we get

$$\sum_{j=0}^{n}(-1)^{n-j}\omega_{m,\alpha}^{(0)}(n,j)(m\hat{h})^j = m^n(\hat{h})_{n,\alpha} = \sum_{j=0}^{n}S_{1,\alpha}(n,j)m^{n-j}(m\hat{h})^j. \tag{2.8}$$

Comparing the coefficients of both side of (2.8), we have

$$(-1)^{n-j}\omega_{m,\alpha}^{(0)}(n,j) = m^{n-j}S_{1,\alpha}(n,j). \tag{2.9}$$

In addition, from (1.33) and Theorem 2.1 (ii), we get

$$\begin{aligned} \sum_{j=0}^{n}W_{m,\alpha}^{(0)}(n,j)m^j(\hat{h})_{j,\alpha} &= (m\hat{n})^n = m^n(\hat{h})^n \\ &= m^n\sum_{j=0}^{n}S_{2,\alpha}(n,j)(\hat{h})_{j,\alpha}. \end{aligned} \tag{2.10}$$

By comparing the coefficients of both side of (2.10), we have

$$W^{(0)}_{m,\alpha}(n,j) = m^{n-j} S_{2,\alpha}(n,j). \tag{2.11}$$

■

For $r \in \mathbb{N} \cup \{0\}$, the unsigned r-Stirling numbers of the first kind are given by

$$\langle x+r \rangle_n = \sum_{k=0}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r x^k, \quad (n \geq 0), \quad (\text{see } [21, 23]). \tag{2.12}$$

From (2.12), we consider the α-analogues of unsigned r-Stirling numbers defined by

$$\langle x+r \rangle_{n,\alpha} = \sum_{k=0}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\alpha} x^k, \quad (n \geq 0). \tag{2.13}$$

From (2.13), we observe that

$$\begin{aligned} (x-r)_{n,\alpha} &= (-1)^n \langle -x+r \rangle_{n,\alpha} = (-1)^n \sum_{k=0}^{n} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\alpha} (-x)^k \\ &= \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\alpha} x^k. \end{aligned} \tag{2.14}$$

By replacing x by $x+r$ in (2.14), we have

$$(x)_{n,\alpha} = \sum_{k=0}^{n} (-1)^{n-k} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_{r,\alpha} (x+r)^k. \tag{2.15}$$

For any polynomial $f(x) = \sum_{n=0}^{\infty} a_n x_n$ by (1.27) and (2.15), we get

$$\left(x\frac{d}{dx} \right)_{k,\alpha} f(x) = \sum_{j=0}^{k} (-1)^{k-j} \begin{bmatrix} k+r \\ j+r \end{bmatrix}_{r,\alpha} \left(x\frac{d}{dx} + r \right)^k \tag{2.16}$$

or equivalently,

$$(\hat{h})_{k,\alpha} = \sum_{j=0}^{k} (-1)^{k-j} \begin{bmatrix} k+r \\ j+r \end{bmatrix}_{r,\alpha} (\hat{h}+r)^k. \tag{2.17}$$

Theorem 2.3. *For $n,\ j \geq 0$ with $n \geq j$, we have*

$$\omega^{(r)}_{1,\alpha}(n,j) = \begin{bmatrix} n+r \\ j+r \end{bmatrix}_{r,\alpha} \quad \text{and} \quad W^{(r)}_{1,\alpha}(n,j) = S^{(r)}_{2,\alpha}(n+r, j+r).$$

Proof. By Theorem 2.2, (1.31) and (2.17), when $m = 1$, we have

$$\sum_{j=0}^{n} (-1)^{n-j} \omega^{(r)}_{1,\alpha}(n,j)(\hat{h}+r)^j = (\hat{h})_{n,\alpha} = \sum_{j=0}^{n} (-1)^{n-j} \begin{bmatrix} n+r \\ j+r \end{bmatrix}_{r,\alpha} (\hat{h}+r)^n \tag{2.18}$$

and

$$\sum_{j=0}^{n} W^{(r)}_{1,\alpha}(n,j)(\hat{h})_{j,\alpha} = (\hat{h}+r)^n = \sum_{j=0}^{n} S^{(r)}_{2,\alpha}(n+r, j+r)(\hat{h})_{j,\alpha}. \tag{2.19}$$

Comparing the coefficients of both sides of (2.18) and (2.19), respectively, we have the desired result.

■

Theorem 2.4. *For $n,\ j \geq 0$ with $n \geq j$, we have*

$$(i) \quad \omega_{m,\alpha}^{(r)}(n,j) = \sum_{l=j}^{n} \binom{l}{j} m^{n-l}(-1)^{n+l} r^{l-j} S_{1,\alpha}(n,l).$$

The inverse formula of (i) is

$$(ii) \quad W_{m,\alpha}^{(r)}(n,j) = \sum_{l=j}^{n} \binom{n}{l} m^{l-j} r^{n-l} S_{2,\alpha}(l,j).$$

Proof. From (1.28) and Theorem 2.1 (i), we have

$$\begin{aligned}
\sum_{j=0}^{n}(-1)^{n-j}\omega_{m,\alpha}^{(r)}(n,j)(m\hat{h}+r)^j &= m^n(\hat{h})_{n,\alpha} \\
&= m^n \sum_{l=0}^{n} S_{1,\alpha}(n,l)(\hat{h})^l \\
&= m^n \sum_{l=0}^{n} S_{1,\alpha}(n,l)\frac{1}{m^l}(m\hat{h}+r-r)^l \\
&= m^n \sum_{l=0}^{n} S_{1,\alpha}(n,l)\frac{1}{m^l}\sum_{j=0}^{l}\binom{l}{j}(m\hat{h}+r)^j(-r)^{l-j} \\
&= \sum_{j=0}^{n}\left(\sum_{l=j}^{n}\binom{l}{j}m^{n-l}(-r)^{l-j}S_{1,\alpha}(n,l)\right)(m\hat{h}+r)^j.
\end{aligned} \tag{2.20}$$

Comparing the coefficients of both sides of (2.20), we have

$$(-1)^{n-j}\omega_{m,\alpha}^{(r)}(n,j) = \sum_{l=j}^{n}\binom{l}{j}m^{n-l}(-r)^{l-j}S_{1,\alpha}(n,l). \tag{2.21}$$

From (1.31) and Theorem 2.1 (ii), we observe that

$$\begin{aligned}
\sum_{j=0}^{n} W_{m,\alpha}^{(r)}(n,j)m^j(\hat{h})_{j,\alpha} = (m\hat{h}+r)^n &= \sum_{l=0}^{n}\binom{n}{l}m^l(\hat{h})^l r^{n-l} \\
&= \sum_{l=0}^{n}\binom{n}{l}m^l r^{n-l}\sum_{j=0}^{l}S_{2,\alpha}(l,j)(\hat{h})_{j,\alpha} \\
&= \sum_{j=0}^{n}\left(\sum_{l=j}^{n}\binom{n}{l}m^l r^{n-l}S_{2,\alpha}(l,j)\right)(\hat{h})_{j,\alpha}.
\end{aligned} \tag{2.22}$$

By comparing the coefficients of both sides of (2.22), we have

$$W_{m,\alpha}^{(r)}(n,j) = \sum_{l=j}^{n}\binom{n}{l}m^{l-j}r^{n-l}S_{2,\alpha}(l,j). \tag{2.23}$$

■

By (1.11) and (2.12), we note that

$$\begin{bmatrix} n \\ j \end{bmatrix}_{\alpha} = (-1)^{n-j}S_{1,\alpha}(n,j), \qquad (n \geq j \geq 0). \tag{2.24}$$

When $m = 1$, by Theorem 2.4, we get the following corollary.

Corollary 2.5. *For $n,\ j \geq 0$ with $n \geq j$, we have*

$$\omega_{1,\alpha}^{(r)}(n,j) = \begin{bmatrix} k+r \\ j+r \end{bmatrix}_{r,\alpha} = \sum_{l=j}^{n} \binom{l}{j} (-1)^{n-l} r^{l-j} S_{1,\alpha}(n,l)$$

and

$$W_{1,\alpha}^{(r)}(n,j) = S_{2,\alpha}^{(r)}(n+r, j+r) = \sum_{l=j}^{n} \binom{n}{l} r^{n-l} S_{2,\alpha}(l,j).$$

Theorem 2.6. *For $n,\ j \geq 0$ with $n \geq j$, we have*

$$\omega_{m,\alpha}^{(r)}(n+1,j) = \omega_{m,\alpha}^{(r)}(n,j-1) + (mn\alpha + r)\omega_{m,\alpha}^{(r)}(n,j)$$

and

$$W_{m,\alpha}^{(r)}(n+1,j) = W_{m,\alpha}^{(r)}(n,j-1) + (mj\alpha + r)W_{m,\alpha}^{(r)}(n,j).$$

Proof. From Theorem 2.1 (i), we have

$$\begin{aligned}
&\sum_{j=0}^{n+1} (-1)^{n+1-j} \omega_{m,\alpha}^{(r)}(n+1,j)(m\hat{h}+r)^j = m^{n+1}(\hat{h})_{n+1,\alpha} \\
&= m^{n+1}(\hat{h})_{n,\alpha}\left(\frac{m\hat{h} - mn\alpha + r - r}{m}\right) \\
&= \sum_{j=0}^{n} (-1)^{n-j} \omega_{m,\alpha}^{(r)}(n,j)(m\hat{h}+r)^{j+1} - (mn\alpha + r)\sum_{j=0}^{n} (-1)^{n-j} \omega_{m,\alpha}^{(r)}(n,j)(m\hat{h}+r)^j \\
&= \sum_{j=0}^{n+1} (-1)^{n+1-j} \omega_{m,\alpha}^{(r)}(n,j-1)(m\hat{h}+r)^j - (mn\alpha + r)\sum_{j=0}^{n+1} (-1)^{n-j} \omega_{m,\alpha}^{(r)}(n,j)(m\hat{h}+r)^j \\
&= \sum_{j=0}^{n+1} (-1)^{n+1-j} \left\{ \omega_{m,\alpha}^{(r)}(n,j-1) + (mn\alpha + r)\sum_{j=0}^{n+1} \omega_{m,\alpha}^{(r)}(n,j) \right\} (m\hat{h}+r)^j.
\end{aligned} \tag{2.25}$$

Comparing the coefficients of both side of (2.25), we have

$$\omega_{m,\alpha}^{(r)}(n+1,j) = \omega_{m,\alpha}^{(r)}(n,j-1) - (mn\alpha + r)\omega_{m,\alpha}^{(r)}(n,j). \tag{2.26}$$

By Theorem 2.1 (ii), we get

$$\begin{aligned}
&\sum_{j=0}^{n+1} W_{m,\alpha}^{(r)}(n+1,j) m^j (\hat{h})_{j,\alpha} = (m\hat{h}+r)^{n+1} = (m\hat{h}+r)^n (m\hat{h}+r) \\
&= (m\hat{h}+r)^n (m\hat{h} - mj\alpha + mj\alpha + r) \\
&= \sum_{j=0}^{n} W_{m,\alpha}^{(r)}(n,j) m^{j+1} (\hat{h})_{j+1,\alpha} + (mj\alpha + r)\sum_{j=0}^{n} W_{m,\alpha}^{(r)}(n,j) m^j (\hat{h})_{j,\alpha} \\
&= \sum_{j=1}^{n+1} W_{m,\alpha}^{(r)}(n,j-1) m^j (\hat{h})_{j,\alpha} + (mj\alpha + r)\sum_{j=0}^{n} W_{m,\alpha}^{(r)}(n,j) m^j (\hat{h})_{j,\alpha} \\
&= \sum_{j=0}^{n+1} \left\{ W_{m,\alpha}^{(r)}(n,j-1) + (mj\alpha + r) W_{m,\alpha}^{(r)}(n,j) \right\} m^j (\hat{h})_{j,\alpha}.
\end{aligned} \tag{2.27}$$

Comparing the coefficients of both side of (2.27), we have

$$W_{m,\alpha}^{(r)}(n+1,j) = W_{m,\alpha}^{(r)}(n,j-1) + (mj\alpha + r)W_{m,\alpha}^{(r)}(n,j). \tag{2.28}$$

■

Corollary 2.7. *For $n \geq 0$, we have*

$$\omega_{m,\alpha}^{(r)}(n,0) = (-1)^n \prod_{l=0}^{n-1}(ml\alpha + r) \quad \textit{and} \quad W_{m,\alpha}^{(r)}(n,0) = r^n.$$

Proof. From (2.26) in Theorem 2.6, we note that

$$\begin{aligned}\omega_{m,\alpha}^{(r)}(n+1,0) &= -(mn\alpha + r)\omega_{m,\alpha}^{(r)}(n,0)\\ &= (-1)^2(mn\alpha + r)(m(n-1)\alpha + r)\omega_{m,\alpha}^{(r)}(n-1,0)\\ &= \cdots\\ &= (-1)^n(mn\alpha + r)(m(n-1)\alpha + r)\cdots r\omega_{m,\alpha}^{(r)}(0,0).\end{aligned} \tag{2.29}$$

By (2.29), we have

$$\omega_{m,\alpha}^{(r)}(n,0) = (-1)^n \prod_{l=0}^{n-1}(ml\alpha + r). \tag{2.30}$$

From (2.28) in Theorem 2.6, we observe that

$$W_{m,\alpha}^{(r)}(n+1,0) = rW_{m,\alpha}^{(r)}(n,0) = r^2 W_{m,\alpha}^{(r)}(n-1,0) = \cdots = r^{n+1}W_{m,\alpha}^{(r)}(0,0). \tag{2.31}$$

By (2.31), we have

$$W_{m,\alpha}^{(r)}(n,0) = r^n. \tag{2.32}$$

■

Theorem 2.8. *For $n,\ l,\ j \in \mathbb{N} \cup \{0\}$, we have*

$$\omega_{m,\alpha}^{(r+1)}(n,l) = \sum_{j=l}^{n}\binom{j}{l}(-1)^{l-j}\omega_{m,\alpha}^{(r)}(n,j) \quad \textit{and} \quad W_{m,\alpha}^{(r+1)}(n,j) = \sum_{l=j}^{n}\binom{n}{l}W_{m,\alpha}^{(r)}(l,j).$$

Proof. From Theorem 2.1 (i), we have

$$\begin{aligned}m^n(\hat{h})_{n,\alpha} &= \sum_{j=0}^{n}(-1)^{n-j}\omega_{m,\alpha}^{(r)}(n,j)(m\hat{h} + r)^j\\ &= \sum_{j=0}^{n}(-1)^{n-j}\omega_{m,\alpha}^{(r)}(n,j)(m\hat{h} + r + 1 - 1)^j\\ &= \sum_{j=0}^{n}(-1)^{n-j}\omega_{m,\alpha}^{(r)}(n,j)\sum_{l=0}^{j}\binom{j}{l}(m\hat{h} + r + 1)^l\\ &= \sum_{l=0}^{n}\left\{\sum_{j=l}^{n}\binom{j}{l}(-1)^{n-j}\omega_{m,\alpha}^{(r)}(n,j)\right\}(m\hat{h} + r + 1)^l.\end{aligned} \tag{2.33}$$

On the other hand, by Theorem 2.1 (i), we have

$$m^n(\hat{h})_{n,a} = \sum_{l=0}^{n}(-1)^{n-l}\omega_{m,\alpha}^{(r+1)}(n,l)(m\hat{h}+r+1)^l. \tag{2.34}$$

By (2.33) and (2.34), we have

$$\omega_{m,\alpha}^{(r+1)}(n,l) = \sum_{j=l}^{n}\binom{j}{l}\omega_{m,\alpha}^{(r)}(n,j). \tag{2.35}$$

From Theorem 2.1 (ii), we have

$$\begin{aligned}\sum_{j=0}^{n} W_{m,\alpha}^{(r+1)}(n,j)m^j(\hat{h})_{j,\alpha} &= (m\hat{h}+r+1)^n \\ &= \sum_{l=0}^{n}\binom{n}{l}(m\hat{h}+r)^l \\ &= \sum_{l=0}^{n}\binom{n}{l}\sum_{j=0}^{l} W_{m,\alpha}^{(r)}(l,j)m^j(\hat{h})_{j,\alpha} \\ &= \sum_{j=0}^{n}\left\{\sum_{l=j}^{n}\binom{n}{l}W_{m,\alpha}^{(r)}(l,j)\right\}m^j(\hat{h})_{j,\alpha}.\end{aligned} \tag{2.36}$$

Comparing the coefficients of both sides of (2.36), we have

$$W_{m,\alpha}^{(r+1)}(n,j) = \sum_{l=j}^{n}\binom{n}{l}W_{m,\alpha}^{(r)}(l,j). \tag{2.37}$$

■

We recall that the coherent state

$$|z\rangle = e^{-\frac{|z|^2}{z}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle, \quad \text{(see [14, 15, 17])}, \tag{2.38}$$

where z is an arbitrary complex constant $a|z\rangle = z|z\rangle = 1$ and $\langle z|z\rangle = 1$.
For $x,\ y \in \mathbb{C}$, we note that

$$\langle x|y\rangle = e^{-\frac{1}{2}(|x|^2+|y|^y)} + \bar{x}y = e^{-\frac{|x|^2}{2}-\frac{|y|^2}{2}}\sum_{n=0}^{\infty}\frac{(\bar{x}y)^n}{n!}, \quad \text{(see [14, 15, 17])}. \tag{2.39}$$

By using the properties of coherent state, from (1.13) and (1.28),

$$\langle z|(\hat{h})^n|z\rangle = bel_n(|z|^2). \tag{2.40}$$

It is easy to show that

$$\begin{aligned}a^\dagger a e^{(a^\dagger a-\alpha+r)t} &= e^{(a^\dagger a+r-\alpha)t}a^\dagger a \\ &= a^\dagger e^{(aa^\dagger-\alpha+r)t}a = a^\dagger e^{(a^\dagger a+1+r-\alpha)t}a, \quad \text{(see [8, 17])}\end{aligned} \tag{2.41}$$

Theorem 2.9. *For $n \geq 0$, we have*

$$\sum_{j=0}^{n}\sum_{l=0}^{j} W_{m,\alpha}^{(r)}(n,j)m^j S_{1,\alpha}(j,l)bel_l(|z|^2) = |z|^{2n}(me^t-1)^n.$$

Proof. Let $f(t) = \langle\, z \mid e^{t(m\hat{h}+r)} \mid z\,\rangle = \langle\, z \mid e^{t(ma^\dagger a+r)} \mid z\,\rangle$. Then, by (2.41), we observe that

$$\begin{aligned}\frac{\partial f(t)}{\partial t} &= \langle\, z \mid (ma^\dagger a+r)e^{t(ma^\dagger a+r)} \mid z\,\rangle \\ &= \langle\, z \mid ma^\dagger e^{t(ma^\dagger a+1+r)}a \mid z\,\rangle + r\langle\, z \mid e^{t(ma^\dagger a+r)} \mid z\,\rangle \\ &= me^t\bar{z}z\langle\, z \mid e^{t(ma^\dagger a+r)} \mid z\,\rangle + r\langle\, z \mid e^{t(ma^\dagger a+1+r)} \mid z\,\rangle \\ &= \big(me^t|z|^2+r\big)f(t).\end{aligned} \tag{2.42}$$

Note that $f(0) = 1$. From (2.42), we have

$$\log f(t) = \int_0^t \frac{f'(t)}{f(t)}dt = \int_0^t (|z|^2me^t+r)dt = |z|^2(me^t-1)+rt. \tag{2.43}$$

Thus, (1.15) and (2.43), we get

$$f(t) = \exp\Bigg(|z|^2(me^t-1)+rt\Bigg) = e^{rt}e^{|z|^2(me^t-1)} \tag{2.44}$$

where $\exp(t) = e^t$.
On the other hand, from (1.29), (2.7) and (2.40), we note that

$$\begin{aligned}f(t) = \langle\, z \mid e^{t(m\hat{h}+r)} \mid z\,\rangle &= \sum_{n=0}^{\infty}\langle z|(m\hat{h}+r)^n|z\rangle\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n} W_{m,\alpha}^{(r)}(n,j)m^j\langle z|(\hat{h})_{j,\alpha}|z\rangle\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n} W_{m,\alpha}^{(r)}(n,j)m^j\sum_{l=0}^{j} S_{1,\alpha}(j,l)\langle z|(\hat{h})^l|z\rangle\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{l=0}^{j} W_{m,\alpha}^{(r)}(n,j)m^j S_{1,\alpha}(j,l)bel_l(|z|^2)\frac{t^n}{n!}.\end{aligned} \tag{2.45}$$

By comparing the coefficients of (2.44) and (2.45), we have the desired identity.
■

Theorem 2.10. *For $k,\ n \geq 0$, we have*

$$\begin{aligned}\langle z|m^n(\hat{h})_{n,\alpha}|z\rangle &= \sum_{q=0}^{n}\sum_{j=l}^{n}\sum_{l=q}^{j}\binom{j}{l}m^l r^{j-l}(-1)^{n-j}\omega_{m,\alpha}^{(r)}(n,j)S_2(l,q)|z|^{2q} \\ &= m^n e^{-|z|^2}\sum_{k=0}^{\infty}(k)_{n,\alpha}\frac{|z|^{2k}}{k!}.\end{aligned}$$

Proof. By Theorem 2.1 (i), we have

$$\begin{aligned}
\langle z|m^n(\hat{h})_{n,\alpha}|z\rangle &= \sum_{j=0}^{n}(-1)^{n-j}\omega_{m,\alpha}^{(r)}(n,j)\langle z|(m\hat{h}+r)^j|z\rangle \\
&= \sum_{j=0}^{n}(-1)^{n-j}\omega_{m,\alpha}^{(r)}(n,j)\sum_{l=0}^{j}\binom{j}{l}m^l\langle z|(\hat{h})^l|z\rangle r^{j-l} \\
&= \sum_{j=0}^{n}(-1)^{n-j}\omega_{m,\alpha}^{(r)}(n,j)\sum_{l=0}^{j}\binom{j}{l}m^l r^{j-l}\sum_{q=0}^{l}S_2(l,q)|z|^{2q} \\
&= \sum_{q=0}^{n}\sum_{j=l}^{n}\sum_{l=q}^{j}\binom{j}{l}m^l r^{j-l}(-1)^{n-j}\omega_{m,\alpha}^{(r)}(n,j)S_2(l,q)|z|^{2q}.
\end{aligned} \tag{2.46}$$

On the other hand, we observe that

$$\begin{aligned}
\langle z|m^n(\hat{h})_{n,\alpha}|z\rangle &= m^n\langle z|(a^\dagger a)_{n,\alpha}|z\rangle \\
&= m^n\sum_{k,l=0}^{\infty}\frac{z^k\bar{z}^l}{\sqrt{k!}\sqrt{l!}}\langle l|k\rangle(k)_{n,\alpha}e^{-\frac{|z|^2}{z}}e^{-\frac{|\bar{z}|^2}{z}} \\
&= m^n e^{-|z|^2}\sum_{k=0}^{\infty}(k)_{n,\alpha}\frac{|z|^{2k}}{k!}.
\end{aligned} \tag{2.47}$$

From (2.46) and (2.47), we have the desired identity.
■

3 Further Remark

In view of (1.13), we consider the α-analogues of r-Dowling polynomials of the first kind

$$d_{m,\alpha}^{(r)}(n,x) = \sum_{k=0}^{n}(-1)^{n-k}\omega_{m,\alpha}^{(r)}(n,k)x^k, \qquad (n \geq 0). \tag{3.1}$$

When $x = 1$, $d_{m,\alpha}^{(r)}(n,1) = d_{m,\alpha}^{(r)}(n)$ are called the α-analogues of r-Dowling numbers of the first kind.

By Proposition 2.2 and (3.1), we get the generating function of the α-analogues of r-Dowling numbers of the first kind as follows.

Proposition 3.1. *For $n \geq 0$, we have*

$$\sum_{n=0}^{\infty}d_{m,\alpha}^{(r)}(n,x)\frac{t^n}{n!} = e_{\alpha m}^{-r}(t)\exp\left(\frac{\log(1+\alpha mt)}{\alpha m}\right),$$

where $e_{\alpha m}^{-r}(t) = (1+\alpha mt)^{-\frac{r}{\alpha m}}$ and $\exp(t) = e^t$.

In view of (1.13), we define the α-analogues of r-Dowling polynomials of the second kind as

$$D_{m,\alpha}^{(r)}(n,x) = \sum_{k=0}^{n}W_{m,\alpha}^{(r)}(n,k)x^k, \qquad (n \ \geq \ 0). \tag{3.2}$$

when $x = 1, D_{m,\alpha}^{(r)}(n,1) = D_{m,\alpha}^{(r)}(n)$ are called the α-analogue r-Dowling numbers of the second kind.

By Proposition 2.1 and (3.2), we have the generating function of r-Dowling polynomials of the second kind as follows.

Proposition 3.2. *For $n \geq 0$, we have*

$$e^{rt} \exp\left(\frac{e^{\alpha m t}-1}{\alpha m}\right) = \sum_{n=0}^{\infty} D_{m,\alpha}^{(r)}(n,x)\frac{t^n}{n!},$$

where $\exp(t) = e^t$.

4 Conclusions

We studied various combinatorial properties of the the α-analogues of r-Whitney of the first and those of second kind arising from the algebraic properties of the number operators. In Theorem 2.1 (i), the α-analogues integral power of number operator $(\hat{n})_{n,\alpha}$ is expressed by the finite sum of the integral power of number operator $(m\hat{n}+r)^n$ with the coefficients the α-analogues of r-Whitney numbers of the first kind in Theorem 2.1 and in Theorem 2.1 (ii) an integral power of number operator $m\hat{n}+r$ is expressed by the finite sum of the number operator $(\hat{n})_{n,\alpha}$ with the coefficients the α-analogues of r-Whitney numbers of the second kind. We obtained properties when $r = 0$ in Theorem 2.2, when $m = 1$ in Theorem 2.3, and relations between the α-analogues Strling numbers and these new numbers in Theorem 2.4. We showed recurrence relations in Theorem 2.6 and 2.8, and interesting identities by using the coherent state in Theorem 2.9 and 2.10. In further remark, the degenerate r-Dowling polynomials are considered as a natural extension of the α-analogues of r-Whitney numbers of the first and those of second kind. For future projects, we would like to conduct research into some potential applications of the special numbers and polynomials (for example, α-analogue r-Dowling polynomials of the first and second kind in Proposition 3.1 and 3.2 in further remark).

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