Some Fundamental Properties of Variational Kurzweil-Henstock-Stieltjes Integral on a Compact Interval in $\mathbb{R}^n$

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

Hoffman introduced the variational Kurzweil-Hentock-Stieltjes integral on a real-valued function and presented some of its properties. In this paper, we defined the variational Kurzweil-Henstock-Stieltjes integral on a compact interval in $\mathbb{R}^n$. Fundamental properties such as uniqueness, linearity property and monotonocity property of both the integrand and integrator, additivity and integrability over a subinterval are provided. In addition, a characterization of the variational Kurzweil-Henstock-Stieltjes integral is established by formulating the Cauchy Criterion.

1   Introduction

In the past centuries, Isaac Newton and Gottfried Wilhelm Leibniz started the modern theory of integration. In 1660's, Newton coined the term "calculus" as he created the theory of fluxions and invented the method of inverse tangents to solve areas under the curves [1]. During 1680's, Leibniz

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discovered the reversal process for finding the tangent lines to solve areas [2]. Both of them found out that integration, being a process of summation, is inverse to the operation of differentiation. This discovery paved way for many applications to mechanics, physics, and other fields. Meanwhile, Bernhard Riemann introduced the Riemann integral by separating the notion of integration from differentiation through limiting process for solving areas. He considered all functions on an interval for which this process of integration could be defined as the class of integrable functions.

However, it was discovered during the nineteenth century that the Riemann integral had many shortcomings [3]. In 1902, Henry Lebesgue introduced the Lebesgue integral to overcome the drawbacks of Riemann integral [4]. But, learning Lebesgue integration needs the notion of measure theory making it a hard one and it had its defects as it cannot integrate highly oscillating functions. In 1950's, Czech mathematician Jaroslav Kurzweil and English mathematician Ralph Henstock independently established another integration process that remedied the inadequacy of the Lebesgue integral. This integration process is called the Kurzweil-Henstock integral or the Generalized Riemann integral that can integrate highly oscillating functions [2]. Then, it was realized that for real-valued functions, this integration process is equivalent to the variational Kurzweil-Henstock integral, which is a corollary of Saks-Henstock Lemma [5].

In the Riemann, Lebesgue, and Kurzweil-Henstock integrals, a given function is integrated with respect to the identity function. However, their are mathematical problems that can be obtained through the extension of the notion of integral towards integrals in which the given function is integrated with respect to a function different from the identity function. This type of integral had been known as the Stieltjes integral, the extension of any integrals with respect to an integrator which does not have to be the identity in general. This concerns many papers due to its applications in functional analysis, theory of distributions, generalized elementary functions, as well as various kinds of generalized differential equations, including dynamic equations on time scales [6]. There are number of papers concerning Henstock-Stieltjes Integral, one of those is the formulation of the PUL-Stieltjes Integral, a Henstock-Stieljes type of definition that utilizes the notion of a partition of unity [7, 8].

In [9], a characterization of the variational Kurzweil-Henstock integral of Banach-Valued functions on the closed interval $[0, 1]$ was presented. Skvortsov and Solodov [10] established the definition of variational Kurzweil-Henstock integrability to a Banach Space valued functions [10]. In [11], a descriptive characterization of the variational Kurzweil-Henstock Stieltjes integral on the real line including its simple properties was provided. This present study generally aims to define variational Kurzweil-Henstock-Stieltjes integral on a compact interval in $\mathbb{R}^n$ and provide some of its basic properties such as uniqueness, linearity property and monotonocity property of both the integrand and integrator, additivity, integrability over a subinterval and the Cauchy Criterion of such integral.

2 Preliminaries

In this section, some terminologies were discussed for a better undertanding of the paper. Throughout, $\mathbb{R}^n$ denotes the n-Euclidean space, $\mathbb{R}^+$ is the set of positive real numbers, $\mathbb{N}$ is the set of natural numbers, $\mathcal{I}_n([\boldsymbol{a}, \boldsymbol{b}])$ is the collection of all compact subintervals of $[\boldsymbol{a}, \boldsymbol{b}]$ and $V([\boldsymbol{u}, \boldsymbol{v}])$ is the collection of all vertices of $[\boldsymbol{u}, \boldsymbol{v}]$.

Definition 2.1. [3] A *compact interval* in $\mathbb{R}^n$ is a set of the form $[\boldsymbol{a}, \boldsymbol{b}] = \prod_{i=1}^n [a_i, b_i]$, where $-\infty < a_i < b_i < \infty$ for $i = 1, \cdots, n$.

Definition 2.2. [3] Two compact intervals $[\boldsymbol{q}, \boldsymbol{r}], [\boldsymbol{s}, \boldsymbol{t}] \in \mathbb{R}^n$ are said to be *non-overlapping* if

$\prod_{i=1}^{n}(q_i, r_i) \bigcap \prod_{i=1}^{n}(s_i, t_i) = \varnothing$, where $\boldsymbol{q} = (q_1, q_2, \cdots, q_n), \boldsymbol{r} = (r_1, r_2, \cdots, r_n), \boldsymbol{s} = (s_1, s_2, \cdots, s_n)$ and $\boldsymbol{t} = (t_1, t_2, \cdots, t_n)$.

Definition 2.3. [12] The *volume of an interval* in $\mathbb{R}^n$ is the product of the lengths of its sides, $vol([\boldsymbol{a}, \boldsymbol{b}]) = \prod_{i=1}^{n}(b_i - a_i)$, whenever $-\infty < a_i < b_i < \infty$ for $i = 1, \cdots, n$.

Definition 2.4. [3] A function $\delta : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}^+$ is called a *gauge* on $[\boldsymbol{a}, \boldsymbol{b}]$.

Definition 2.5. [3] If $\{[\boldsymbol{q}_1, \boldsymbol{r}_1], [\boldsymbol{q}_2, \boldsymbol{r}_2], \cdots, [\boldsymbol{q}_p, \boldsymbol{r}_p]\}$ is a finite collection of pairwise non-overlapping subintervals of $[\boldsymbol{a}, \boldsymbol{b}]$ such that $[\boldsymbol{a}, \boldsymbol{b}] = \bigcup_{k=1}^{p}[\boldsymbol{q_k}, \boldsymbol{r_k}]$, then we say that $\{[\boldsymbol{q}_1, \boldsymbol{r}_1], [\boldsymbol{q}_2, \boldsymbol{r}_2], \cdots, [\boldsymbol{q}_p, \boldsymbol{r}_p]\}$ is a *partition* $\mathcal{D}$ of $[\boldsymbol{a}, \boldsymbol{b}]$. A partition $\mathcal{D}$ of $[\boldsymbol{a}, \boldsymbol{b}]$ is a *net* if for any $k \in \{1, 2, \cdots, p\}$, there is a partition $\mathcal{D}_k$ of $[a_k, b_k]$ such that

$$\mathcal{D} = \{ \prod_{k=1}^{p}[u_k, v_k] : [u_k, v_k] \in \mathcal{D}_k \text{ for } k = 1, 2, \cdots, p\}.$$

Definition 2.6. [3] A *point-interval pair* $(\boldsymbol{t}, \boldsymbol{I})$ consists of a point $\boldsymbol{t} \in \mathbb{R}^n$ and an interval $\boldsymbol{I} \in \mathbb{R}^n$. Here, $\boldsymbol{t}$ is a *tag* of $\boldsymbol{I}$.

Definition 2.7. [3] For each $\boldsymbol{x} \in \mathbb{R}^n$, we define $||| \cdot |||$ the *maximum norm* of $\boldsymbol{x}$ by

$$|||\boldsymbol{x}||| = \max\{|x_i| : i = 1, \cdots, n\}, \text{ where } \boldsymbol{x} = (x_1, \cdots, x_n).$$

Definition 2.8. [3] Given $\boldsymbol{x} \in \mathbb{R}^n$ and $r > 0$, we set $B(\boldsymbol{x}, r) = \{\boldsymbol{y} \in \mathbb{R}^n : |||\boldsymbol{x} - \boldsymbol{y}||| < r\}$, where $\boldsymbol{x} - \boldsymbol{y} = (x_1 - y_1, \cdots, x_n - y_n)$, $\boldsymbol{x} = (x_1, \cdots, x_n)$, and $\boldsymbol{y} = (y_1, \cdots, y_n)$.

Definition 2.9. [3] A *Perron partition* $\boldsymbol{P}$ of $[\boldsymbol{a}, \boldsymbol{b}]$ is a finite collection of point-interval pairs $\{(\boldsymbol{t}_k, [\boldsymbol{q}_k, \boldsymbol{r}_k]) : k = 1, 2, \cdots, p\}$ where $\{[\boldsymbol{q}_1, \boldsymbol{r}_1], [\boldsymbol{q}_2, \boldsymbol{r}_2], \cdots, [\boldsymbol{q}_p, \boldsymbol{r}_p]\}$ is a partition of $[\boldsymbol{a}, \boldsymbol{b}]$ and $\boldsymbol{t}_k \in [\boldsymbol{q}_k, \boldsymbol{r}_k]$ for $k = 1, 2, \cdots, p$.

Definition 2.10. [3] Given a gauge δ defined on $\{\boldsymbol{t}_1, \cdots, \boldsymbol{t}_p\}$, the Perron partition $\boldsymbol{P}$ of $[\boldsymbol{a}, \boldsymbol{b}]$ is said to be $\boldsymbol{\delta}$-*fine* if $[\boldsymbol{q}_k, \boldsymbol{r}_k] \subseteq B(\boldsymbol{t}_k, \delta(\boldsymbol{t}_k))$ for $k = 1, 2, \cdots, p$.

Lemma 2.1. [3] *(Cousin's Lemma)* *If δ is a gauge on $[\boldsymbol{a}, \boldsymbol{b}]$, then there exists a δ-fine Perron partition of $[\boldsymbol{a}, \boldsymbol{b}]$.*

Definition 2.11. [3] A finite collection $\{(\boldsymbol{t}_k, [\boldsymbol{q}_k, \boldsymbol{r}_k]) : k = 1, \cdots, p\}$ of point-interval pair is said to be *Perron subpartition* of $[\boldsymbol{a}, \boldsymbol{b}]$ if $\boldsymbol{t}_k \in [\boldsymbol{q}_k, \boldsymbol{r}_k]$ for $k = 1, 2, \cdots, p$ and $\{[\boldsymbol{q}_1, \boldsymbol{r}_1], [\boldsymbol{q}_2, \boldsymbol{r}_2], \cdots, [\boldsymbol{q}_p, \boldsymbol{r}_p]\}$ is a finite collection of non-overlapping intervals in $[\boldsymbol{a}, \boldsymbol{b}]$.

For brevity, we denote $\{(\boldsymbol{t}_k, [\boldsymbol{q}_k, \boldsymbol{r}_k]) : k = 1, \cdots, p\}$ by $\{(\boldsymbol{t}, [\boldsymbol{q}, \boldsymbol{r}])\}$.

Definition 2.12. [3] Given a gauge δ defined on $\{\boldsymbol{t}_1, \cdots, \boldsymbol{t}_p\}$, the Perron subpartition $\boldsymbol{S}$ of $[\boldsymbol{a}, \boldsymbol{b}]$ is said to be $\boldsymbol{\delta}$-*fine* if $[\boldsymbol{q}_k, \boldsymbol{r}_k] \subseteq B(\boldsymbol{t}_k, \delta(\boldsymbol{t}_k))$ for $k = 1, 2, \cdots, p$.

Lemma 2.2. [3] *(Saks-Henstock Lemma)* If $f \in \boldsymbol{KH}[\boldsymbol{a}, \boldsymbol{b}]$, then for every $\varepsilon > 0$ there exists a gauge δ on $[\boldsymbol{a}, \boldsymbol{b}]$ such that

$$\sum_{(\boldsymbol{t}, [\boldsymbol{q}, \boldsymbol{r}]) \in \boldsymbol{S}} \big| f(\boldsymbol{t})vol([\boldsymbol{q}, \boldsymbol{r}]) - (\boldsymbol{KH}) \int_{[\boldsymbol{q}, \boldsymbol{r}]} f \big| < \varepsilon$$

for each δ-fine Perron subpartition $S = \{(\boldsymbol{t}, [\boldsymbol{q}, \boldsymbol{r}])\}$ of $[\boldsymbol{a}, \boldsymbol{b}]$.

Definition 2.13. [13] A real-valued set function F defined on a class of sets $\mathscr{I}$ is called *additive* (or finitely additive) if

$$F\left(\bigcup_{i=1}^{k} J_i\right) = \sum_{i=1}^{k} F(J_i)$$

for all $k \in \mathbb{N}$ and all disjoint sets $J_1, J_2, \cdots, J_k \in \mathscr{I}$ such that $\bigcup_{i=1}^{k} J_i \in \mathscr{I}$.

Definition 2.14. [3] Let $g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$. The *total variation of* $\boldsymbol{g}$ over $[\boldsymbol{a}, \boldsymbol{b}]$ is given by

$$Var(g, [\boldsymbol{a}, \boldsymbol{b}]) = sup\left\{ \sum_{[\boldsymbol{u}, \boldsymbol{v}] \in \mathcal{D}} |\Delta_g([\boldsymbol{u}, \boldsymbol{v}])| : \mathcal{D} \text{ is a partition of } [\boldsymbol{a}, \boldsymbol{b}] \right\}$$

where

$$\Delta_g([\boldsymbol{u}, \boldsymbol{v}]) = \sum_{\boldsymbol{t} \in V([\boldsymbol{u}, \boldsymbol{v}])} \left(g(\boldsymbol{t}) \prod_{k=1}^{n} (-1)^{\chi_{\{u_k\}}(t_k)} \right) \qquad (\star)$$

and $[\boldsymbol{u}, \boldsymbol{v}] \in \mathcal{I}_n([\boldsymbol{a}, \boldsymbol{b}])$.

Example 2.3. When $n = 1$, $(\star)$ becomes

$$\Delta_g([u_1, v_1]) = g(u_1)(-1)^{\chi_{\{u_1\}}(u_1)} + g(v_1)(-1)^{\chi_{\{u_1\}}(v_1)} = g(v_1) - g(u_1).$$

Theorem 2.4. [3] *If* $\{\boldsymbol{I}_1, \boldsymbol{I}_2, \cdots, \boldsymbol{I}_p\} \subset \mathcal{I}_n([\boldsymbol{a}, \boldsymbol{b}])$ *is a finite collection of non-overlapping intervals in* $\mathbb{R}^n$, *then there exists a net* D_0 *and* $\boldsymbol{J} \cap \boldsymbol{I}_r \in \mathcal{I}_n([\boldsymbol{a}, \boldsymbol{b}])$ *for some* $r \in \{1, 2, \cdots, p\}$ *such that* $\boldsymbol{J} \subseteq \boldsymbol{I}_r$.

3 Main Results

This part finally introduces the variational Kurzweil-Henstock-Stieltjes integral on $[\boldsymbol{a}, \boldsymbol{b}] \subseteq \mathbb{R}^n$ and presents some of its basic properties.

Definition 3.1. Let $f, g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ be functions. The function f is said to be *variational Kurzweil-Henstock-Stieltjes integrable* (or simply $\boldsymbol{vKHS}$-integrable) with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$, if there exists an additive function $F : \mathcal{I} \to \mathbb{R}$, where $\mathcal{I}$ is the set of all compact subintervals of $[\boldsymbol{a}, \boldsymbol{b}]$ satisfying the following condition: Given $\varepsilon > 0$, there exists a gauge δ such that

$$\sum_{(\boldsymbol{t}, [\boldsymbol{u}, \boldsymbol{v}]) \in \boldsymbol{P}} \left| f(\boldsymbol{t}) \Delta_g([\boldsymbol{u}, \boldsymbol{v}]) - F([\boldsymbol{u}, \boldsymbol{v}]) \right| < \varepsilon$$

for every δ-fine Perron partition $\boldsymbol{P} = \{(\boldsymbol{t}, [\boldsymbol{u}, \boldsymbol{v}])\}$ of $[\boldsymbol{a}, \boldsymbol{b}]$. Here, if such additive function F exists, we write $F([\boldsymbol{u}, \boldsymbol{v}]) = \displaystyle\int_{[\boldsymbol{u}, \boldsymbol{v}]} f \, dg$, for all $[\boldsymbol{u}, \boldsymbol{v}] \in \mathcal{I}$ and we say F is the *indefinite* $\boldsymbol{vKHS}$-**integral** of f with respect to g.

Throughout this paper, denote by $\boldsymbol{vKHS}([\boldsymbol{a}, \boldsymbol{b}], g)$ the set of functions $f : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ which are $\boldsymbol{vKHS}$-integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$.

Example 3.1. Let $\mathbb{Q}$ be the set of all rational numbers, and define the function $f : [0, 1] \to \mathbb{R}$ by setting

$$f(x) = \begin{cases} 1, & x \in [0, 1] \cap \mathbb{Q} \\ 0, & x \in [0, 1] \setminus \mathbb{Q}. \end{cases}$$

and $g(x) = x$. We show that f is $\boldsymbol{vKHS}$-integrable with respect to g.

Note that $\mathbb{Q}$ is a countable set. Put $\langle r_n \rangle_{n=1}^{\infty}$ as an enumeration of $[0,1] \cap \mathbb{Q}$. Fix $\varepsilon > 0$. Define a gauge δ on $[0,1]$ by setting

$$\delta(x) = \begin{cases} \dfrac{\varepsilon}{2^{n+1}}, & x = r_n \text{ for some } n \in \mathbb{N} \\ 1, & x \in [0,1] \setminus \mathbb{Q}. \end{cases}$$

Let P be a δ-fine Perron partition of $[0,1]$, then

$$\sum_{(t,[u,v]) \in P} |f(t)\Delta_g([u,v]) - 0| = \sum_{(t,[u,v]) \in P} |f(t)[g(v) - g(u)] - 0|$$

$$= \sum_{(t,[u,v]) \in P} |f(t)(v-u)|$$

$$= \sum_{\substack{(t,[u,v]) \in P \\ t \in \mathbb{Q} \cap [0,1]}} |f(t)(v-u)| + \sum_{\substack{(t,[u,v] \in P) \\ t \in [0,1] \setminus \mathbb{Q}}} |f(t)(v-u)|$$

$$= \sum_{\substack{(t,[u,v]) \in P \\ t \in \mathbb{Q} \cap [0,1]}} |f(t)(v-u)|$$

$$= \sum_{\substack{(t,[u,v]) \in P \\ t \in \mathbb{Q} \cap [0,1]}} (v-u)$$

$$< \sum_{\substack{(t,[u,v]) \in P \\ t \in \mathbb{Q} \cap [0,1]}} 2\delta(t)$$

$$= \sum_{\substack{(t,[u,v]) \in P \\ t \in \mathbb{Q} \cap [0,1]}} 2\left(\frac{\varepsilon}{2^{n+1}}\right)$$

$$< 2\varepsilon \sum_{\substack{(t,[u,v]) \in P \\ t \in \mathbb{Q} \cap [0,1]}} \left(\frac{1}{2^{n+1}}\right)$$

$$= \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $f \in \boldsymbol{vKHS}[0,1]$ and $F([0,1]) = \displaystyle\int_{[0,1]} f \, dg = 0$.

Now, we will provide some fundamental properties of $\boldsymbol{vKHS}$-integral.

Theorem 3.2. *(Uniqueness) There is at most one value satisfying the Definition 3.1.*

Proof: Let $F, G : \mathcal{I} \to \mathbb{R}$, where $\mathcal{I}$ is the set of all compact subintervals in $[\boldsymbol{a}, \boldsymbol{b}]$ be additive functions. Suppose $F(\boldsymbol{J})$ and $G(\boldsymbol{K})$ for all $\boldsymbol{J}, \boldsymbol{K} \in \mathcal{I}$ be the values of the $\boldsymbol{vKHS}$-integral with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$ such that Definition 3.1 holds. Fix $\varepsilon > 0$. Since both $F(\boldsymbol{J})$ and $G(\boldsymbol{K})$ for all $\boldsymbol{J}, \boldsymbol{K} \in \mathcal{I}$ satisfy Definition 3.1, there exist gauges, $\boldsymbol{\delta}_1$ and $\boldsymbol{\delta}_2$ on $[\boldsymbol{a}, \boldsymbol{b}]$ respectively, such that

$$\sum_{(\boldsymbol{t},\boldsymbol{J}) \in \boldsymbol{P}_1} \left| f(\boldsymbol{t})\Delta_g(\boldsymbol{J}) - F(\boldsymbol{J}) \right| < \frac{\varepsilon}{2}$$

for every $\boldsymbol{\delta}_1$-fine Perron partition $\boldsymbol{P}_1 = \{(\boldsymbol{t}, \boldsymbol{J})\}$ of $[\boldsymbol{a}, \boldsymbol{b}]$ and

$$\sum_{(\boldsymbol{s},\boldsymbol{K}) \in \boldsymbol{P}_2} \left| f(\boldsymbol{s})\Delta_g(\boldsymbol{K}) - G(\boldsymbol{K}) \right| < \frac{\varepsilon}{2}$$

for every $\boldsymbol{\delta}_2$-fine Perron partition $\boldsymbol{P}_2 = \{(\boldsymbol{s}, \boldsymbol{K})\}$ of $[\boldsymbol{a}, \boldsymbol{b}]$. Define $\boldsymbol{\delta}$ by setting

$$\boldsymbol{\delta}(\boldsymbol{x}) = min\{\boldsymbol{\delta}_1(\boldsymbol{x}), \boldsymbol{\delta}_2(\boldsymbol{x})\} \tag{3.1}$$

for all $\boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}]$. Then $\boldsymbol{\delta}$ is a gauge on $[\boldsymbol{a}, \boldsymbol{b}]$. By Cousin's Lemma, we can fix a δ-fine Perron partition $\boldsymbol{P}$ of $[\boldsymbol{a}, \boldsymbol{b}]$. Equation (3.1) implies that $\boldsymbol{P}$ is both $\boldsymbol{\delta}_1$-fine and $\boldsymbol{\delta}_2$-fine. Denote $\mathscr{C} = \{\boldsymbol{I} : (\boldsymbol{\xi}, \boldsymbol{I}) \in \boldsymbol{P}\}$. We claim that $F(\boldsymbol{I}) = G(\boldsymbol{I})$ for all $\boldsymbol{I} \in \mathscr{C}$. Let $\boldsymbol{J} \in \mathscr{C}$. Observe that

$$\begin{aligned}
\sum_{(\boldsymbol{x}, \boldsymbol{J}) \in \boldsymbol{P}} |F(\boldsymbol{J}) - G(\boldsymbol{J})| &= \sum_{(\boldsymbol{x}, \boldsymbol{J}) \in \boldsymbol{P}} \left|F(\boldsymbol{J}) - f(\boldsymbol{x})\Delta_g(\boldsymbol{J}) + f(\boldsymbol{x})\Delta_g(\boldsymbol{J}) - G(\boldsymbol{J})\right| \\
&\leq \sum_{(\boldsymbol{x}, \boldsymbol{J}) \in \boldsymbol{P}} \left\{ \left|F(\boldsymbol{J}) - f(\boldsymbol{x})\Delta_g(\boldsymbol{J})\right| + \left|f(\boldsymbol{x})\Delta_g(\boldsymbol{J}) - G(\boldsymbol{J})\right| \right\} \\
&= \sum_{(\boldsymbol{x}, \boldsymbol{J}) \in \boldsymbol{P}} \left|f(\boldsymbol{x})\Delta_g(\boldsymbol{J}) - F(\boldsymbol{J})\right| + \sum_{(\boldsymbol{x}, \boldsymbol{J}) \in \boldsymbol{P}} \left|f(\boldsymbol{x})\Delta_g(\boldsymbol{J}) - G(\boldsymbol{J})\right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $\displaystyle\sum_{(\boldsymbol{x}, \boldsymbol{J}) \in \boldsymbol{P}} |F(\boldsymbol{J}) - G(\boldsymbol{J})| = 0$. Hence, $|F(\boldsymbol{J}) - G(\boldsymbol{J})| = 0$ for all $\boldsymbol{J} \in \mathscr{C}$. Thus, $F(\boldsymbol{J}) = G(\boldsymbol{J})$ for all $\boldsymbol{J} \in \mathscr{C}$. $\qquad\square$

Theorem 3.3. (Linearity of Integrand) *Let $g, f_1, f_2 : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ be functions. If $f_1, f_2 \in \boldsymbol{vKHS}([\boldsymbol{a}, \boldsymbol{b}], g)$, then for any $\alpha, \beta \in \mathbb{R}$, $\alpha f_1 + \beta f_2 \in \boldsymbol{vKHS}([\boldsymbol{a}, \boldsymbol{b}], g)$ and*

$$\int_{[\boldsymbol{a}, \boldsymbol{b}]} (\alpha f_1 + \beta f_2) \, dg = \alpha \int_{[\boldsymbol{a}, \boldsymbol{b}]} f_1 \, dg + \beta \int_{[\boldsymbol{a}, \boldsymbol{b}]} f_2 \, dg.$$

Proof: Let $g, f_1, f_2 : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$ and $\varepsilon > 0$ be given. Suppose that f_1 and f_2 are $\boldsymbol{vKHS}$-integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$. Put $F = \int_{[\boldsymbol{a}, \boldsymbol{b}]} f_1 \, dg$ and $G = \int_{[\boldsymbol{a}, \boldsymbol{b}]} f_2 \, dg$. Then there is a gauge δ_1 on $[\boldsymbol{a}, \boldsymbol{b}]$ such that

$$\sum_{(\boldsymbol{t}, \boldsymbol{J}) \in \boldsymbol{P}_1} \left|f_1(\boldsymbol{t})\Delta_g(\boldsymbol{J}) - F(\boldsymbol{J})\right| < \frac{\varepsilon}{2(|\alpha| + 1)}$$

for every δ_1-fine Perron partition $\boldsymbol{P}_1 = \{(\boldsymbol{t}, \boldsymbol{J})\}$ on $[\boldsymbol{a}, \boldsymbol{b}]$. In the same manner, there is a gauge δ_2 on $[\boldsymbol{a}, \boldsymbol{b}]$ such that

$$\sum_{(\boldsymbol{s}, \boldsymbol{K}) \in \boldsymbol{P}_2} \left|f_2(\boldsymbol{s})\Delta_g(\boldsymbol{K}) - G(\boldsymbol{K})\right| < \frac{\varepsilon}{2(|\beta| + 1)}$$

for every δ_2-fine Perron partition $\boldsymbol{P}_2 = \{(\boldsymbol{s}, \boldsymbol{K})\}$ on $[\boldsymbol{a}, \boldsymbol{b}]$. Define δ by setting

$$\delta(\boldsymbol{x}) = \min\{\delta_1(\boldsymbol{x}), \delta_2(\boldsymbol{x})\} \tag{3.2}$$

for all $\boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}]$. Then δ is a gauge on $[\boldsymbol{a}, \boldsymbol{b}]$. Hence, we can choose a δ-fine Perron partition $\boldsymbol{P}$ of

$[\boldsymbol{a}, \boldsymbol{b}]$. By (3.2), $\boldsymbol{P}$ is both δ_1-fine and δ_2-fine. Consequently,

$$\sum_{(\boldsymbol{t},\boldsymbol{J})\in \boldsymbol{P}} \big|[\alpha f_1(\boldsymbol{t}) + \beta f_2(\boldsymbol{t})]\Delta_g(\boldsymbol{J}) - \{\alpha F(\boldsymbol{J}) + \beta G(\boldsymbol{J})\}\big|$$

$$= \sum_{(\boldsymbol{t},\boldsymbol{J})\in \boldsymbol{P}} \big|\alpha\{[f_1(\boldsymbol{t})]\Delta_g(\boldsymbol{J}) - F(\boldsymbol{J})\} + \beta\{[f_2(\boldsymbol{t})]\Delta_g(\boldsymbol{J}) - G(\boldsymbol{J})\}\big|$$

$$\leq \sum_{(\boldsymbol{t},\boldsymbol{J})\in \boldsymbol{P}} \left\{ \big|\alpha\{[f_1(\boldsymbol{t})]\Delta_g(\boldsymbol{J}) - F(\boldsymbol{J})\}\big| + \big|\beta\{[f_2(\boldsymbol{t})]\Delta_g(\boldsymbol{J}) - G(\boldsymbol{J})\}\big| \right\}$$

$$= |\alpha| \sum_{(\boldsymbol{t},\boldsymbol{J})\in \boldsymbol{P}} |f_1(\boldsymbol{t})\Delta_g(\boldsymbol{J}) - F(\boldsymbol{J})| + |\beta| \sum_{(\boldsymbol{t},\boldsymbol{J})\in \boldsymbol{P}} |f_2(\boldsymbol{t})\Delta_g(\boldsymbol{J}) - G(\boldsymbol{J})|$$

$$< |\alpha|\frac{\varepsilon}{2(|\alpha| + 1)} + |\beta|\frac{\varepsilon}{2(|\beta| + 1)}$$

$$< (|\alpha| + 1)\frac{\varepsilon}{2(|\alpha| + 1)} + (|\beta| + 1)\frac{\varepsilon}{2(|\beta| + 1)}$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon.$$

Thus,

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} (\alpha f_1 + \beta f_2)\ dg = \alpha \int_{[\boldsymbol{a},\boldsymbol{b}]} f_1\ dg + \beta \int_{[\boldsymbol{a},\boldsymbol{b}]} f_2\ dg. \qquad \square$$

Proposition 3.1. Let $g_1, g_2 : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ be functions. Then for any $\alpha, \beta \in \mathbb{R}$,

$$\Delta_{\alpha g_1 + \beta g_2}([\boldsymbol{u}, \boldsymbol{v}]) = \alpha\Delta_{g_1}([\boldsymbol{u}, \boldsymbol{v}]) + \beta\Delta_{g_2}([\boldsymbol{u}, \boldsymbol{v}]),\ \text{where } [\boldsymbol{u}, \boldsymbol{v}] \in \mathcal{I}_n\big([\boldsymbol{a}, \boldsymbol{b}]\big).$$

Proof: The proof is obvious. $\qquad \square$

Theorem 3.4. (Linearity of Integrator) *Let $f, g_1, g_2 : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ be functions. Suppose $f \in \boldsymbol{vKHS}([\boldsymbol{a}, \boldsymbol{b}], g_1) \cap \boldsymbol{vKHS}([\boldsymbol{a}, \boldsymbol{b}], g_2)$. Then for any $\alpha, \beta \in \mathbb{R}$, $f \in \boldsymbol{vKHS}([\boldsymbol{a}, \boldsymbol{b}], \alpha g_1 + \beta g_2)$ and*

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ d[\alpha g_1 + \beta g_2] = \alpha \int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg_1 + \beta \int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg_2.$$

Proof: Using the same arguments in the proof of Theorem 3.3 and by Proposition 3.1 the result follows. $\qquad \square$

Theorem 3.5. Let $f_1, f_2, g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ be functions. If $f_1, f_2 \in \boldsymbol{vKHS}([\boldsymbol{a}, \boldsymbol{b}], g)$ and $f_1(\boldsymbol{x}) \leq f_2(\boldsymbol{x})$ for all $\boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}]$, then

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} f_1\ dg \leq \int_{[\boldsymbol{a},\boldsymbol{b}]} f_2\ dg.$$

Proof: Let $f_1, f_2, g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ and $\varepsilon > 0$. Assume that f_1 and f_2 are $\boldsymbol{vKHS}$-integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$ and $f_1(\boldsymbol{x}) \leq f_2(\boldsymbol{x})$ for all $\boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}]$. Denote $F = \int_{[\boldsymbol{a},\boldsymbol{b}]} f_1\ dg$ and $G = \int_{[\boldsymbol{a},\boldsymbol{b}]} f_2\ dg$. Then there exists a gauge δ_1 on $[\boldsymbol{a}, \boldsymbol{b}]$ such that

$$\sum_{(\boldsymbol{t},\boldsymbol{J})\in \boldsymbol{P}_1} \Big|f_1(\boldsymbol{t})\Delta_g(\boldsymbol{J}) - F(\boldsymbol{J})\Big| < \frac{\varepsilon}{2}$$

for every δ_1-fine Perron partition $\boldsymbol{P}_1 = \{(\boldsymbol{t}, \boldsymbol{J})\}$ on $[\boldsymbol{a}, \boldsymbol{b}]$. In the same manner, there exists a gauge δ_2 on $[\boldsymbol{a}, \boldsymbol{b}]$ such that

$$\sum_{(\boldsymbol{s},\boldsymbol{K})\in \boldsymbol{P}_2} \Big|f_2(\boldsymbol{s})\Delta_g(\boldsymbol{K}) - G(\boldsymbol{K})\Big| < \frac{\varepsilon}{2}$$

for every δ_2-fine Perron partition $\boldsymbol{P_2} = \{(\boldsymbol{s}, \boldsymbol{K})\}$ on $[\boldsymbol{a}, \boldsymbol{b}]$. Define δ by setting

$$\delta(\boldsymbol{x}) = \min\{\delta_1(\boldsymbol{x}), \delta_2(\boldsymbol{x})\} \tag{3.3}$$

for all $\boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}]$. Then δ is a gauge on $[\boldsymbol{a}, \boldsymbol{b}]$. Let $\boldsymbol{P}$ be a δ-fine Perron partition of $[\boldsymbol{a}, \boldsymbol{b}]$. By (3.3), $\boldsymbol{P}$ is both δ_1-fine and δ_2-fine. Notice that

$$\sum_{(\boldsymbol{x}, \boldsymbol{H}) \in \boldsymbol{P}} f_1(\boldsymbol{x}) \Delta_g(\boldsymbol{H}) \leq \sum_{(\boldsymbol{x}, \boldsymbol{H}) \in \boldsymbol{P}} f_2(\boldsymbol{x}) \Delta_g(\boldsymbol{H}).$$

Now,

$$\frac{\varepsilon}{2} > \sum_{(\boldsymbol{x}, \boldsymbol{H}) \in \boldsymbol{P}} \left| f_1(\boldsymbol{x}) \Delta_g(\boldsymbol{H}) - F(\boldsymbol{H}) \right| \geq \sum_{(\boldsymbol{x}, \boldsymbol{H}) \in \boldsymbol{P}} \{ F(\boldsymbol{H}) - f_1(\boldsymbol{x}) \Delta_g(\boldsymbol{H}) \}.$$

This indicates that

$$\frac{\varepsilon}{2} > \sum_{(\boldsymbol{x}, \boldsymbol{H}) \in \boldsymbol{P}} \{ F(\boldsymbol{H}) - f_1(\boldsymbol{x}) \Delta_g(\boldsymbol{H}) \} = \sum_{(\boldsymbol{x}, \boldsymbol{H}) \in \boldsymbol{P}} F(\boldsymbol{H}) - \sum_{(\boldsymbol{x}, \boldsymbol{H}) \in \boldsymbol{P}} f_1(\boldsymbol{x}) \Delta_g(\boldsymbol{H}).$$

So,

$$\sum_{(\boldsymbol{x}, \boldsymbol{H}) \in \boldsymbol{P}} F(\boldsymbol{H}) < \sum_{(\boldsymbol{x}, \boldsymbol{H}) \in \boldsymbol{P}} f_1(\boldsymbol{x}) \Delta_g(\boldsymbol{H}) + \frac{\varepsilon}{2}. \tag{3.4}$$

Similarly,

$$\frac{\varepsilon}{2} > \sum_{(\boldsymbol{x}, \boldsymbol{H}) \in \boldsymbol{P}} \left| f_2(\boldsymbol{x}) \Delta_g(\boldsymbol{H}) - G(\boldsymbol{H}) \right| \geq \sum_{(\boldsymbol{x}, \boldsymbol{H}) \in \boldsymbol{P}} \{ f_2(\boldsymbol{x}) \Delta_g(\boldsymbol{H}) - G(\boldsymbol{H}) \}.$$

This would mean that

$$\frac{\varepsilon}{2} > \sum_{(\boldsymbol{x}, \boldsymbol{H}) \in \boldsymbol{P}} \{ f_2(\boldsymbol{x}) \Delta_g(\boldsymbol{H}) - G(\boldsymbol{H}) \} = \sum_{(\boldsymbol{x}, \boldsymbol{H}) \in \boldsymbol{P}} f_2(\boldsymbol{x}) \Delta_g(\boldsymbol{H}) - \sum_{(\boldsymbol{x}, \boldsymbol{H}) \in \boldsymbol{P}} G(\boldsymbol{H}).$$

And so,

$$\sum_{(\boldsymbol{x}, \boldsymbol{H}) \in \boldsymbol{P}} f_2(\boldsymbol{x}) \Delta_g(\boldsymbol{H}) + \frac{\varepsilon}{2} < \sum_{(\boldsymbol{x}, \boldsymbol{H}) \in \boldsymbol{P}} G(\boldsymbol{H}) + \varepsilon. \tag{3.5}$$

By (3.4) and (3.5),

$$\sum_{(\boldsymbol{x}, \boldsymbol{H}) \in \boldsymbol{P}} F(\boldsymbol{H}) < \sum_{(\boldsymbol{x}, \boldsymbol{H}) \in \boldsymbol{P}} f_1(\boldsymbol{x}) \Delta_g(\boldsymbol{H}) + \frac{\varepsilon}{2} \leq \sum_{(\boldsymbol{x}, \boldsymbol{H}) \in \boldsymbol{P}} f_2(\boldsymbol{x}) \Delta_g(\boldsymbol{H}) + \frac{\varepsilon}{2}$$

$$< \sum_{(\boldsymbol{x}, \boldsymbol{H}) \in \boldsymbol{P}} G(\boldsymbol{H}) + \varepsilon.$$

Thus,

$$\int_{[\boldsymbol{a}, \boldsymbol{b}]} f_1 \ dg \leq \int_{[\boldsymbol{a}, \boldsymbol{b}]} f_2 \ dg. \qquad \qquad \Box$$

Proposition 3.2. Let $g_1, g_2 : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ be functions. If $g_1(\boldsymbol{t}) \leq g_2(\boldsymbol{t})$ for all $\boldsymbol{t} \in [\boldsymbol{u}, \boldsymbol{v}]$, where $[\boldsymbol{u}, \boldsymbol{v}] \in \mathcal{I}_n \big([\boldsymbol{a}, \boldsymbol{b}]\big)$, then $\Delta_{g_1}([\boldsymbol{u}, \boldsymbol{v}]) \leq \Delta_{g_2}([\boldsymbol{u}, \boldsymbol{v}])$.

Proof: The proof is obvious. $\qquad \qquad \Box$

Theorem 3.6. Let $f, g_1, g_2 : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ be functions. If $f \in \boldsymbol{vKHS}([\boldsymbol{a}, \boldsymbol{b}], g_1) \cap \boldsymbol{vKHS}([\boldsymbol{a}, \boldsymbol{b}], g_2)$ and $g_1(\boldsymbol{x}) \leq g_2(\boldsymbol{x})$ for all $\boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}]$, then

$$\int_{[\boldsymbol{a}, \boldsymbol{b}]} f \ dg_1 \leq \int_{[\boldsymbol{a}, \boldsymbol{b}]} f \ dg_2.$$

Proof: The proof is similar to that of Theorem 3.5 and by Proposition 3.2, the theorem holds. □

Theorem 3.7. (Cauchy Criterion) *Let f, g be real-valued functions defined on $[\boldsymbol{a}, \boldsymbol{b}]$. Then $f \in \boldsymbol{vKHS}([\boldsymbol{a}, \boldsymbol{b}], g)$ if and only if for every $\varepsilon > 0$, there exists a gauge δ on $[\boldsymbol{a}, \boldsymbol{b}]$ such that*

$$\left| \sum_{(\boldsymbol{t}, \boldsymbol{J}) \in P} f(\boldsymbol{t}) \Delta_g(\boldsymbol{J}) - \sum_{(\boldsymbol{s}, \boldsymbol{K}) \in Q} f(\boldsymbol{s}) \Delta_g(\boldsymbol{K}) \right| < \varepsilon$$

whenever $P = \{(\boldsymbol{t}, \boldsymbol{J})\}$ and $Q = \{(\boldsymbol{s}, \boldsymbol{K})\}$ are δ-fine Perron partitions of $[\boldsymbol{a}, \boldsymbol{b}]$.

Proof: ($\Rightarrow$) Let $f, g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ and $\varepsilon > 0$. Assume that f is $\boldsymbol{vKHS}$-integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$. Then there exists an additive function $F = \int_{[\boldsymbol{a}, \boldsymbol{b}]} f \, dg$ and a gauge δ on $[\boldsymbol{a}, \boldsymbol{b}]$ such that

$$\sum_{(\boldsymbol{t}, \boldsymbol{J}) \in P_0} \left| f(\boldsymbol{t}) \Delta_g(\boldsymbol{J}) - F(\boldsymbol{J}) \right| < \frac{\varepsilon}{2} \tag{3.6}$$

for every δ-fine Perron partition $P_0 = \{(\boldsymbol{t}, \boldsymbol{J})\}$ on $[\boldsymbol{a}, \boldsymbol{b}]$. Let $P = \{(\boldsymbol{t}, \boldsymbol{J})\}$ and $Q = \{(\boldsymbol{s}, \boldsymbol{K})\}$ be any δ-fine Perron partitions of $[\boldsymbol{a}, \boldsymbol{b}]$. By (3.6),

$$\left| \sum_{(\boldsymbol{t}, \boldsymbol{J}) \in P} f(\boldsymbol{t}) \Delta_g(\boldsymbol{J}) - \sum_{(\boldsymbol{s}, \boldsymbol{K}) \in Q} f(\boldsymbol{s}) \Delta_g(\boldsymbol{K}) \right| < \varepsilon.$$

($\Leftarrow$) Suppose for each $n \in \mathbb{N}$ there exists a gauge δ_n on $[\boldsymbol{a}, \boldsymbol{b}]$ such that

$$\left| \sum_{(\boldsymbol{t}, \boldsymbol{J}) \in P_n} f(\boldsymbol{t}) \Delta_g(\boldsymbol{J}) - \sum_{(\boldsymbol{s}, \boldsymbol{K}) \in Q_n} f(\boldsymbol{s}) \Delta_g(\boldsymbol{K}) \right| < \frac{1}{n}$$

for every δ_n-fine Perron partitions, $P_n = \{(\boldsymbol{t}, \boldsymbol{J})\}$ and $Q_n = \{(\boldsymbol{s}, \boldsymbol{K})\}$ of $[\boldsymbol{a}, \boldsymbol{b}]$. Define a gauge $\delta_n^\star$ on $[\boldsymbol{a}, \boldsymbol{b}]$ by

$$\delta_n^\star(\boldsymbol{x}) = \min\{\delta_1(\boldsymbol{x}), \delta_2(\boldsymbol{x}), \cdots, \delta_n(\boldsymbol{x})\} \tag{3.7}$$

for all $\boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}]$. By (3.7), we can fix a $\delta_n^\star$-fine Perron partition of $[\boldsymbol{a}, \boldsymbol{b}]$. We claim that the sequence, $\left\langle \sum_{(\boldsymbol{t}, \boldsymbol{J}) \in P_n} f(\boldsymbol{t}) \Delta_g(\boldsymbol{J}) \right\rangle_{n=1}^{\infty}$ is a Cauchy sequence of real numbers. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. If n_1 and n_2 are natural numbers that satisfy $\min\{n_1, n_2\} \geq N$, then P_{n_1} and P_{n_2} are both $\delta_{\min\{n_1, n_2\}}^\star$-fine Perron partitions of $[\boldsymbol{a}, \boldsymbol{b}]$. Hence,

$$\left| \sum_{(\boldsymbol{t}, \boldsymbol{J}) \in P_{n_1}} f(\boldsymbol{t}) \Delta_g(\boldsymbol{J}) - \sum_{(\boldsymbol{s}, \boldsymbol{K}) \in P_{n_2}} f(\boldsymbol{s}) \Delta_g(\boldsymbol{K}) \right| < \frac{1}{\min\{n_1, n_2\}} \leq \frac{1}{N}.$$

This means that $\left\langle \sum_{(\boldsymbol{t}, \boldsymbol{J}) \in P_n} f(\boldsymbol{t}) \Delta_g(\boldsymbol{J}) \right\rangle_{n=1}^{\infty}$ is a Cauchy sequence of real numbers. Then the sequence $\left\langle \sum_{(\boldsymbol{t}, \boldsymbol{J}) \in P_n} f(\boldsymbol{t}) \Delta_g(\boldsymbol{J}) \right\rangle_{n=1}^{\infty}$ converges to, say, $F \in \mathbb{R}$. Next, the remaining part is to prove that f is $\boldsymbol{vKHS}$-integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$ and $F = \int_{[\boldsymbol{a}, \boldsymbol{b}]} f \, dg$. Let P be a $\delta_N^\star$-fine Perron partition of $[\boldsymbol{a}, \boldsymbol{b}]$. Notice that the sequence, $\langle \delta_n^\star \rangle_{n=1}^{\infty}$ of gauges is non-increasing since $\delta_n^\star \geq \delta_{n+1}^\star$ for all $n \in \mathbb{N}$, then P_n is $\delta_N^\star$-fine for every natural number $n \geq N$. Consequently,

$$\left| \sum_{(\boldsymbol{t}, \boldsymbol{J}) \in P} f(\boldsymbol{t}) \Delta_g(\boldsymbol{J}) - F \right| = \left| \sum_{(\boldsymbol{t}, \boldsymbol{J}) \in P} f(\boldsymbol{t}) \Delta_g(\boldsymbol{J}) - \lim_{n \to \infty} \sum_{(\boldsymbol{t}, \boldsymbol{J}) \in P_n} f(\boldsymbol{t}) \Delta_g(\boldsymbol{J}) \right|$$

$$= \lim_{n \to \infty} \left| \sum_{(\boldsymbol{t}, \boldsymbol{J}) \in P} f(\boldsymbol{t}) \Delta_g(\boldsymbol{J}) - \sum_{(\boldsymbol{t}, \boldsymbol{J}) \in P_n} f(\boldsymbol{t}) \Delta_g(\boldsymbol{J}) \right|$$

$$< \lim_{n \to \infty} \frac{1}{n} \leq \lim_{n \to \infty} \frac{1}{N} = \frac{1}{N} < \varepsilon.$$

Thus, the theorem holds. □

Theorem 3.8. *Let f, g be real-valued functions defined on $[\boldsymbol{a}, \boldsymbol{b}]$. If $f \in \boldsymbol{vKHS}([\boldsymbol{a}, \boldsymbol{b}], g)$, then $f \in \boldsymbol{vKHS}(\boldsymbol{I}, g)$ for all $\boldsymbol{I} \in \mathcal{I}_n([\boldsymbol{a}, \boldsymbol{b}])$.*

Proof: Let $f, g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ and $\varepsilon > 0$. Assume that f is $\boldsymbol{vKHS}$-integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$ and $\boldsymbol{I} \in \mathcal{I}_n([\boldsymbol{a}, \boldsymbol{b}])$. By Theorem 3.7, there is a gauge δ on $[\boldsymbol{a}, \boldsymbol{b}]$ such that

$$\left| \sum_{(\boldsymbol{t}, \boldsymbol{J}) \in P} f(\boldsymbol{t}) \Delta_g(\boldsymbol{J}) - \sum_{(\boldsymbol{s}, \boldsymbol{K}) \in Q} f(\boldsymbol{s}) \Delta_g(\boldsymbol{K}) \right| < \varepsilon \tag{3.8}$$

for every δ-fine Perron partitions $P = \{(\boldsymbol{t}, \boldsymbol{J})\}$ and $Q = \{(\boldsymbol{s}, \boldsymbol{K})\}$ of $[\boldsymbol{a}, \boldsymbol{b}]$. Suppose that $\boldsymbol{I} = [\boldsymbol{a}, \boldsymbol{b}]$, then we are done. Assume that $\boldsymbol{I} \subset [\boldsymbol{a}, \boldsymbol{b}]$. Then we can choose a finite collection $\mathcal{H} \subseteq \mathcal{I}_n([\boldsymbol{a}, \boldsymbol{b}])$ such that $\boldsymbol{I} \notin \mathcal{H}$ and $\mathcal{H} \cup \{\boldsymbol{I}\}$ is a net of $[\boldsymbol{a}, \boldsymbol{b}]$. For any $\boldsymbol{H} \in \mathcal{H} \cup \{\boldsymbol{I}\}$, fix a δ-fine Perron partition $P_{\boldsymbol{H}} = \{(\boldsymbol{t}, \boldsymbol{H})\}$ of $\boldsymbol{H}$. Let $P_{\boldsymbol{I}} = \{(\boldsymbol{t}, \boldsymbol{J})\}$ and $Q_{\boldsymbol{I}} = \{(\boldsymbol{s}, \boldsymbol{K})\}$ be δ-fine Perron partitions of $\boldsymbol{I}$. Then

$$P = P_{\boldsymbol{I}} \cup \bigcup_{\boldsymbol{H} \in \mathcal{H}} P_{\boldsymbol{H}} \text{ and } Q = Q_{\boldsymbol{I}} \cup \bigcup_{\boldsymbol{H} \in \mathcal{H}} P_{\boldsymbol{H}}$$

are δ-fine Perron partitions of $[\boldsymbol{a}, \boldsymbol{b}]$. Notice that

$$\sum_{(\boldsymbol{t}, \boldsymbol{J}) \in P} f(\boldsymbol{t}) \Delta_g(\boldsymbol{J}) = \sum_{(\boldsymbol{t}, \boldsymbol{J}) \in P_{\boldsymbol{I}}} f(\boldsymbol{t}) \Delta_g(\boldsymbol{J}) + \sum_{\boldsymbol{H} \in \mathcal{H}} \left[\sum_{(\boldsymbol{t}, \boldsymbol{L}) \in P_{\boldsymbol{H}}} f(\boldsymbol{t}) \Delta_g(\boldsymbol{L}) \right]$$

and

$$\sum_{(\boldsymbol{s}, \boldsymbol{K}) \in Q} f(\boldsymbol{s}) \Delta_g(\boldsymbol{K}) = \sum_{(\boldsymbol{s}, \boldsymbol{K}) \in Q_{\boldsymbol{I}}} f(\boldsymbol{s}) \Delta_g(\boldsymbol{K}) + \sum_{\boldsymbol{H} \in \mathcal{H}} \left[\sum_{(\boldsymbol{t}, \boldsymbol{L}) \in P_{\boldsymbol{H}}} f(\boldsymbol{t}) \Delta_g(\boldsymbol{L}) \right].$$

This means that

$$\sum_{(\boldsymbol{t}, \boldsymbol{J}) \in P_{\boldsymbol{I}}} f(\boldsymbol{t}) \Delta_g(\boldsymbol{J}) = \sum_{(\boldsymbol{t}, \boldsymbol{J}) \in P} f(\boldsymbol{t}) \Delta_g(\boldsymbol{J}) - \sum_{\boldsymbol{H} \in \mathcal{H}} \left[\sum_{(\boldsymbol{t}, \boldsymbol{L}) \in P_{\boldsymbol{H}}} f(\boldsymbol{t}) \Delta_g(\boldsymbol{L}) \right]$$

and

$$\sum_{(\boldsymbol{s}, \boldsymbol{K}) \in Q_{\boldsymbol{I}}} f(\boldsymbol{s}) \Delta_g(\boldsymbol{K}) = \sum_{(\boldsymbol{s}, \boldsymbol{K}) \in Q} f(\boldsymbol{s}) \Delta_g(\boldsymbol{K}) - \sum_{\boldsymbol{H} \in \mathcal{H}} \left[\sum_{(\boldsymbol{t}, \boldsymbol{L}) \in P_{\boldsymbol{H}}} f(\boldsymbol{t}) \Delta_g(\boldsymbol{L}) \right].$$

By (3.8), we obtain

$$\left| \sum_{(\boldsymbol{t}, \boldsymbol{J}) \in P_{\boldsymbol{I}}} f(\boldsymbol{t}) \Delta_g(\boldsymbol{J}) - \sum_{(\boldsymbol{s}, \boldsymbol{K}) \in Q_{\boldsymbol{I}}} f(\boldsymbol{s}) \Delta_g(\boldsymbol{K}) \right|$$
$$= \left| \sum_{(\boldsymbol{t}, \boldsymbol{J}) \in P} f(\boldsymbol{t}) \Delta_g(\boldsymbol{J}) - \sum_{(\boldsymbol{s}, \boldsymbol{K}) \in Q} f(\boldsymbol{s}) \Delta_g(\boldsymbol{K}) \right| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the theorem follows. □

Corollary 3.9. *If f is $\boldsymbol{vKHS}$-integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$, then f is $\boldsymbol{vKHS}$-integrable with respect to g on $\boldsymbol{I}$ for all $\boldsymbol{I} \in \mathcal{I}_n([\boldsymbol{a}, \boldsymbol{b}])$ and*

$$\int_{[\boldsymbol{a}, \boldsymbol{b}]} f \cdot \chi_{\boldsymbol{I}} \, dg = \int_{\boldsymbol{I}} f \, dg.$$

Proof: Let $\boldsymbol{I} \in \mathcal{I}_n([\boldsymbol{a}, \boldsymbol{b}])$. Suppose that f is $\boldsymbol{vKHS}$-integrable with respect to g on $[\boldsymbol{a}, \boldsymbol{b}]$. We may assume that $\boldsymbol{I} = [\boldsymbol{a}, \boldsymbol{b}]$ and $\boldsymbol{I} \subset [\boldsymbol{a}, \boldsymbol{b}]$. If $\boldsymbol{I} = [\boldsymbol{a}, \boldsymbol{b}]$, then

$$\int_{\boldsymbol{I}} f \cdot \chi_{\boldsymbol{I}} \, dg = \int_{[\boldsymbol{a}, \boldsymbol{b}]} f \cdot \chi_{[\boldsymbol{a}, \boldsymbol{b}]} \, dg = \int_{[\boldsymbol{a}, \boldsymbol{b}]} f \, dg = \int_{\boldsymbol{I}} f \, dg.$$

And so, we are done. If $\boldsymbol{I} \subset [\boldsymbol{a}, \boldsymbol{b}]$, then f is $\boldsymbol{vKHS}$-integrable with respect to g on $\boldsymbol{I}$. Note that we can write $[\boldsymbol{a}, \boldsymbol{b}] = ([\boldsymbol{a}, \boldsymbol{b}] \smallsetminus \boldsymbol{I}) \cup \boldsymbol{I}$. Hence,

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} f \cdot \chi_{\boldsymbol{I}} \ dg = \int_{[\boldsymbol{a},\boldsymbol{b}]\smallsetminus \boldsymbol{I}} f \cdot \chi_{\boldsymbol{I}} \ dg + \int_{\boldsymbol{I}} f \cdot \chi_{\boldsymbol{I}} \ dg$$

$$= \int_{[\boldsymbol{a},\boldsymbol{b}]\smallsetminus \boldsymbol{I}} f \cdot (0) \ dg + \int_{\boldsymbol{I}} f \cdot (1) \ dg = \int_{\boldsymbol{I}} f \ dg.$$

Therefore,

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} f \cdot \chi_{\boldsymbol{I}} \ dg = \int_{\boldsymbol{I}} f \ dg. \qquad \square$$

.

Theorem 3.10. *(Additivity)* Let $f, g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ and $\boldsymbol{I}, \boldsymbol{J} \in \mathcal{I}_n([\boldsymbol{a}, \boldsymbol{b}])$ that partitions $[\boldsymbol{a}, \boldsymbol{b}]$. Suppose $f \in \boldsymbol{vKHS}(\boldsymbol{I}, g) \ \cap \ \boldsymbol{vKHS}(\boldsymbol{J}, g)$, then $f \in \boldsymbol{vKHS}([\boldsymbol{a}, \boldsymbol{b}], g)$ and

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} f \ dg = \int_{\boldsymbol{I}} f \ dg + \int_{\boldsymbol{J}} f \ dg.$$

Proof: Suppose f is $\boldsymbol{vKHS}$-integrable with respect to g on $\boldsymbol{I}, \boldsymbol{J} \in \mathscr{I}_n([\boldsymbol{a}, \boldsymbol{b}])$ that partitions $[\boldsymbol{a}, \boldsymbol{b}]$. Since f is $\boldsymbol{vKHS}$-integrable on $\boldsymbol{I}$, it follows that there exists an additive function F such that for every $\varepsilon > 0$, we can choose a gauge δ_1 on $\boldsymbol{I}$ so that

$$\sum_{(\boldsymbol{t},\boldsymbol{I}) \in \boldsymbol{P}_1} \big| f(\boldsymbol{t}) \Delta_g(\boldsymbol{I}) - F(\boldsymbol{I}) \big| < \frac{\varepsilon}{2}$$

for every δ_1-fine Perron partition $\boldsymbol{P}_1 = \{(\boldsymbol{t}, \boldsymbol{I})\}$ on $\boldsymbol{I}$. Similarly, since f is $\boldsymbol{vKHS}$-integrable on $\boldsymbol{J}$, it follows that there exists an additive function G such that for every $\varepsilon > 0$, we can choose a gauge δ_2 on $\boldsymbol{J}$ so that

$$\sum_{(\boldsymbol{s},\boldsymbol{J}) \in \boldsymbol{P}_2} \big| f(\boldsymbol{s}) \Delta_g(\boldsymbol{J}) - G(\boldsymbol{J}) \big| < \frac{\varepsilon}{2}$$

for every δ_2-fine Perron partition $\boldsymbol{P}_2 = \{(\boldsymbol{s}, \boldsymbol{J})\}$ on $\boldsymbol{J}$. Then $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ partitions $[\boldsymbol{a}, \boldsymbol{b}]$. For each $\boldsymbol{M} \in \mathcal{I}_n([\boldsymbol{a}, \boldsymbol{b}])$, define

$$\Phi(\boldsymbol{M}) = \begin{cases} F(\boldsymbol{M}), & \text{if } \boldsymbol{M} \subseteq \boldsymbol{I} \\ G(\boldsymbol{M}), & \text{if } \boldsymbol{M} \subseteq \boldsymbol{J} \\ F(\boldsymbol{M} \cap \boldsymbol{I}) + G(\boldsymbol{M} \cap \boldsymbol{J}), & \text{if } \boldsymbol{M} \nsubseteq \boldsymbol{I} \text{ and } \boldsymbol{M} \nsubseteq \boldsymbol{J}. \end{cases}$$

Next, let $\mathcal{A} = \{\boldsymbol{A}_i \subseteq [\boldsymbol{a}, \boldsymbol{b}] : i = 1, 2, \cdots, p\}$ be a collection of non-overlapping subintervals of $[\boldsymbol{a}, \boldsymbol{b}]$. So, we have

$$\bigcup_{\boldsymbol{K} \in \mathcal{A}} \boldsymbol{K} = \left\{ \big(\bigcup_{\boldsymbol{K} \subseteq \boldsymbol{I}} \boldsymbol{K} \big) \bigcup \big(\bigcup_{\boldsymbol{K} \subseteq \boldsymbol{J}} \boldsymbol{K} \big) \bigcup \big[\bigcup_{\boldsymbol{K} \nsubseteq \boldsymbol{I},\boldsymbol{J}} (\boldsymbol{K} \cap \boldsymbol{I}) \cup (\boldsymbol{K} \cap \boldsymbol{J}) \big] \right\}.$$

Notice that

$$\Phi\big(\bigcup_{\boldsymbol{K} \in \mathcal{A}} \boldsymbol{K} \big) = \Phi\left\{ \big(\bigcup_{\boldsymbol{K} \subseteq \boldsymbol{I}} \boldsymbol{K} \big) \bigcup \big(\bigcup_{\boldsymbol{K} \subseteq \boldsymbol{J}} \boldsymbol{K} \big) \bigcup \big[\bigcup_{\boldsymbol{K} \nsubseteq \boldsymbol{I},\boldsymbol{J}} (\boldsymbol{K} \cap \boldsymbol{I}) \cup (\boldsymbol{K} \cap \boldsymbol{J}) \big] \right\}$$

$$= F\big[\big(\bigcup_{\boldsymbol{K} \subseteq \boldsymbol{I}} \boldsymbol{K} \big) \big] + F(\varnothing) + F\big[\bigcup_{\boldsymbol{K} \nsubseteq \boldsymbol{I},\boldsymbol{J}} (\boldsymbol{K} \cap \boldsymbol{I}) \big] + G(\varnothing)$$

$$+ G\big[\big(\bigcup_{\boldsymbol{K} \subseteq \boldsymbol{J}} \boldsymbol{K} \big) \big] + G\big[\bigcup_{\boldsymbol{K} \nsubseteq \boldsymbol{I},\boldsymbol{J}} (\boldsymbol{K} \cap \boldsymbol{J}) \big]$$

$$= \sum_{\boldsymbol{K} \subseteq \boldsymbol{I}} F(\boldsymbol{K}) + \sum_{\boldsymbol{K} \subseteq \boldsymbol{J}} G(\boldsymbol{K}) + \Big\{ \sum_{\boldsymbol{K} \nsubseteq \boldsymbol{I},\boldsymbol{J}} F(\boldsymbol{K} \cap \boldsymbol{I}) + \sum_{\boldsymbol{K} \nsubseteq \boldsymbol{I},\boldsymbol{J}} G(\boldsymbol{K} \cap \boldsymbol{J}) \Big\}$$

$$= \sum_{\boldsymbol{K} \subseteq \boldsymbol{I}} \Phi(\boldsymbol{K}) + \sum_{\boldsymbol{K} \subseteq \boldsymbol{J}} \Phi(\boldsymbol{K}) + \sum_{\boldsymbol{K} \nsubseteq \boldsymbol{I},\boldsymbol{J}} \Phi(\boldsymbol{K}) = \sum_{\boldsymbol{K} \in \mathcal{A}} \Phi(\boldsymbol{K}).$$

Hence, Φ is an additive function. Now, define δ by setting

$$
\delta(\boldsymbol{x}) = \begin{cases} \min\{d(\boldsymbol{x}, \boldsymbol{J}), \delta_1(\boldsymbol{x})\}, & \text{if } \boldsymbol{x} \in \boldsymbol{I} \smallsetminus \boldsymbol{J} \\ \min\{\delta_1(\boldsymbol{x}), \delta_2(\boldsymbol{x})\}, & \text{if } \boldsymbol{x} \in \boldsymbol{I} \cap \boldsymbol{J} \\ \min\{d(\boldsymbol{x}, \boldsymbol{I}), \delta_2(\boldsymbol{x})\}, & \text{if } \boldsymbol{x} \in \boldsymbol{J} \smallsetminus \boldsymbol{I} \end{cases}
$$

for all $\boldsymbol{x} \in [\boldsymbol{a}, \boldsymbol{b}]$. Then δ is a gauge on $[\boldsymbol{a}, \boldsymbol{b}]$. And so, there is a δ-fine Perron partition $\boldsymbol{P} = \{(\boldsymbol{x}, \boldsymbol{H})\}$ on $[\boldsymbol{a}, \boldsymbol{b}]$. Then

$$
\boldsymbol{P_I} = \{(\boldsymbol{x}, \boldsymbol{K}) : \boldsymbol{x} \in \boldsymbol{I}, \boldsymbol{K} = \boldsymbol{I} \cap \boldsymbol{H} \text{ and } vol(\boldsymbol{K}) > 0\}
$$

is a δ-fine Perron partition of $\boldsymbol{I}$, and

$$
\boldsymbol{P_J} = \{(\boldsymbol{x}, \boldsymbol{L}) : \boldsymbol{x} \in \boldsymbol{J}, \boldsymbol{L} = \boldsymbol{J} \cap \boldsymbol{H} \text{ and } vol(\boldsymbol{L}) > 0\}
$$

is a δ-fine Perron partition of $\boldsymbol{J}$. Thus, $\boldsymbol{P_I}$ is δ_1-fine and $\boldsymbol{P_J}$ is δ_2-fine. Consequently,

$$
\sum_{(\boldsymbol{x},\boldsymbol{H})\in\boldsymbol{P}} \big|f(\boldsymbol{x})\Delta_g(\boldsymbol{H}) - \Phi(\boldsymbol{H})\big| = \sum_{(\boldsymbol{x},\boldsymbol{H})\in\boldsymbol{P_I}} \big|f(\boldsymbol{x})\Delta_g(\boldsymbol{H}) - F(\boldsymbol{H})\big|
$$
$$
+ \sum_{(\boldsymbol{x},\boldsymbol{H})\in\boldsymbol{P_J}} \big|f(\boldsymbol{x})\Delta_g(\boldsymbol{H}) - G(\boldsymbol{H})\big|
$$
$$
< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
$$

Since $\varepsilon > 0$ is arbitrary, we can conclude that

$$
\int_{[\boldsymbol{a},\boldsymbol{b}]} f \, dg = \int_{\boldsymbol{I}} f \, dg + \int_{\boldsymbol{J}} f \, dg. \qquad \square
$$

Theorem 3.11. Let $f, g : [\boldsymbol{a}, \boldsymbol{b}] \to \mathbb{R}$ be functions and let P be a partition of $[\boldsymbol{a}, \boldsymbol{b}]$. Suppose $f \in \boldsymbol{vKHS}(\boldsymbol{J}, g)$ for all $\boldsymbol{J} \in P$, then $f \in \boldsymbol{vKHS}([\boldsymbol{a}, \boldsymbol{b}], g)$ and

$$
\int_{[\boldsymbol{a},\boldsymbol{b}]} f \, dg = \sum_{\boldsymbol{J}\in P} \int_{\boldsymbol{J}} f \, dg.
$$

Proof:

Let P be a partition of $[\boldsymbol{a}, \boldsymbol{b}]$. Suppose that f is $\boldsymbol{vKHS}$-integrable with respect to g on $\boldsymbol{J}$ for all $\boldsymbol{J} \in P$. In view of Theorem 3.8 and Theorem 2.4, we may assume that P is a net of $[\boldsymbol{a}, \boldsymbol{b}]$. Applying Additivity theorem repeatedly, we obtain the desired result. $\qquad \square$

4 Conclusion and Recommendation

The obtained results in this paper are pretty much standard. In particular, the uniqueness, linearity and monotonocity of both the integrand and integrator, integrability over a subinterval, Cauchy criterion and additivity of this integral. As a recommendation, results in the literature may serve as a backbone for further related topics such as the formulation of Saks-Henstock Lemma and some convergence theorems.

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Competing Interests

Authors have declared that no competing interests exist.

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