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An Exponentiated Odd Lindley Inverse Exponential Distribution and its Application to Infant Mortality and HIV Transmission Rates in Nigeria

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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ABSTRACT

Recently, researchers have shown much interest in developing new continuous probability distributions by adding one or two parameter(s) to the some existing baseline distributions. This act has been beneficial to the field of statistical theory especially in modeling of real life situations. Also, the exponentiated family as used in developing new distributions is an efficient method proposed and studied for defining more flexible continuous probability distributions for modeling real life data. In this study, the method of exponentiation has been used to develop a new distribution called "Exponentiated odd Lindley inverse exponential distribution". Some properties of the proposed distribution and estimation of its unknown parameters has been done using the

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method of maximum likelihood estimation and its application to real life datasets. The new model has been applied to infant mortality rate and mother-to-child HIV transmission rate. The results of these two applications reveal that the proposed model is a better model compared to the other fitted existing models by some selection information criteria.

Keywords: Odd Lindley inverse exponential distribution; exponentiated family; properties; maximum likelihood estimation; applications.

1. INTRODUCTION

In past years, different probability distributions have been utilized for the purpose of analyzing lifetime situations however it has been discovered that some of these distributions do not efficiently analyze these real life datasets and hence creating a problem in statistical theory and applications.

Consequently, several compound probability distributions have been introduced and studied for modeling most real life situations and these compound distributions are proven to be flexible and more better in statistical theory compared to their standard counterparts ([1]-[12]).

Due to these reasons, different authors have developed different extensions of the exponential distribution and a list of some of these distributions include the following: [13] proposed the odd Lindley inverse exponential distribution, [14] also introduced and studied the Exponential Inverse Exponential distribution, the Kumaraswamy Inverse Exponential distribution was also developed by [15], in a similar way [16] defined and studied the exponentiated generalized Inverse Exponential distribution, a new Lindley-Exponential distribution was proposed and studied by [17], a Lomax-exponential distribution was also developed by [18], a transmuted odd generalized exponential-exponential distribution was studied by [19], [20] derived the transmuted exponential distribution, [21] proposed and studied a transmuted inverse exponential distribution, the odd generalized exponential-exponential distribution was also studied by [22], a transmuted Weibull-exponential distribution was considered and studied by [23] and the Weibull-exponential distribution was proposed by [24].

In line with the above propositions and the need for a flexible extension of the odd Lindley inverse exponential distribution, this paper introduced and studied a new distribution called "exponentiated odd Lindley inverse exponential distribution".

The probability density function (pdf) of the odd Lindley inverse exponential distribution (OLinInExD) according to [13] is defined by

$$g(x)=\frac{\alpha^2\theta x^{-2}e^{-\frac{\theta}{x}}}{(1+\alpha)\left[1-e^{-\frac{\theta}{x}}\right]^3}\exp\left\{-\alpha\left[\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right]\right\} \tag{1}$$

The corresponding cumulative distribution function (cdf) of odd Lindley inverse exponential distribution (OLinInExD) is given by

$$G(x)=1-\frac{\alpha+\left(1-e^{-\frac{\theta}{x}}\right)}{(1+\alpha)\left(1-e^{-\frac{\theta}{x}}\right)}\exp\left\{-\alpha\left[\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right]\right\} \tag{2}$$

where, $x>0,\theta>0,\alpha>0$; α is the shape parameter and θ is a scale parameter [13]. defined and studied some mathematical and statistical properties of the OLinInExD and discovered that it is better than the Lindley, exponential and inverse exponential distributions.

Hence, this article aimed to develop a new continuous distribution called "an Exponentiated Odd Lindley Inverse Exponential distribution (ExpOLinInExD)". The remaining sections of the article are organized as follows: formulation of the new distribution and its graphs is presented in section 2. The derivation of its properties is given in section 3. The estimation of parameters via method of maximum likelihood estimation is done in section 4. Application of the proposed distribution together with other existing ones to infant mortality rate and mother-to-child HIV transmission rate is presented in section 5. In section 6, the summary and conclusion of the study is given.

2. THE EXPONENTIATED ODD LINDLEY INVERSE EXPONENTIAL DISTRIBUTION (EXPOLININEXD)

Following the work of [25], a random variable *X* is said to have an exponentiated form of distribution function if its cdf and pdf are respectively given by;

$$F(x) = [G(x)]^{\lambda} \quad (3)$$

and

$$f(x) = \lambda g(x)[G(x)]^{\lambda-1} \quad (4)$$

where; $x > 0$, and β is the extra shape parameter, $G(x)$ and $g(x)$ are the cdf and pdf of any continuous distribution to be extended respectively.

Putting equation (1) and (2) into equation (3) and (4) and simplifying, we obtain the cdf and pdf of the ExpOLinInExD given in equation (5) and (6) respectively as follows:

$$F(x) = \left[1 - \frac{\alpha + \left(1 - e^{-\frac{\theta}{x}}\right)}{(1+\alpha)\left(1 - e^{-\frac{\theta}{x}}\right)} \exp\left\{-\alpha\left[\frac{e^{-\frac{\theta}{x}}}{1 - e^{-\frac{\theta}{x}}}\right]\right\}\right]^{\lambda} \quad (5)$$

And

$$f(x) = \frac{\alpha^2 \theta \lambda x^{-2} e^{-\frac{\theta}{x}}}{(1+\alpha)\left[1 - e^{-\frac{\theta}{x}}\right]^3} \exp\left\{-\alpha\left[\frac{e^{-\frac{\theta}{x}}}{1 - e^{-\frac{\theta}{x}}}\right]\right\}\left[1 - \frac{\alpha + \left(1 - e^{-\frac{\theta}{x}}\right)}{(1+\alpha)\left(1 - e^{-\frac{\theta}{x}}\right)} \exp\left\{-\alpha\left[\frac{e^{-\frac{\theta}{x}}}{1 - e^{-\frac{\theta}{x}}}\right]\right\}\right]^{\lambda-1} \quad (6)$$

where $x > 0, \alpha > 0, \theta > 0, \beta > 0$, α and λ are the shape parameters and θ is the scale parameter.

Graphs of the pdf and cdf of the ExpOLinInExD using different parameter values are presented in Fig. 1 as follows.

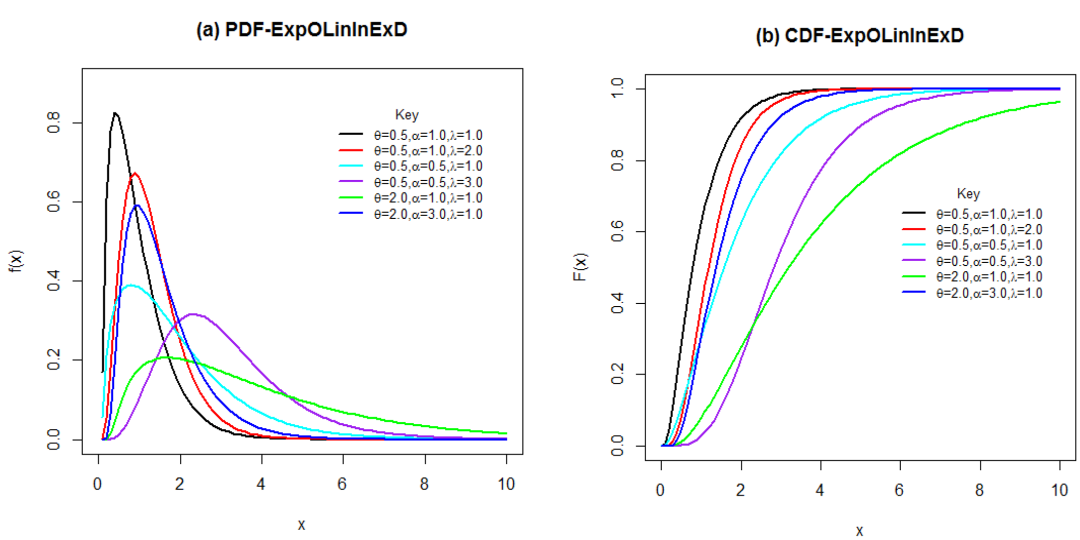

Fig. 1. (a)-PDF and (b)-CDF of the ExpOLinInExD for different values of the parameters

Based on the Fig. 1, it is revealed that the pdf ExpOLinInExD distribution is positively skewed and has different shapes with respect to the parameter values. Again the plot of the cdf depicts that the cdf equals to one when *x* approaches infinity and equals zero when *x* tends to zero as it should be, meaning that it is a valid cdf.

3. STATISTICAL PROPERTIES OF ExpOLinInExD

Section 3 presents useful properties of the ExpOLinInExD distribution. These amongst others include are:

3.1 Moments

Let X denote a continuous random variable, the n^{th} moment of X is given by;

$$\mu_n^{'} = E\left(X^n\right) = \int_0^\infty x^n f(x)dx \tag{7}$$

where f(x) the pdf of the ExpOLinInExD and is stated from (6) as:

$$f(x) = \frac{\alpha^2\theta\lambda x^{-2} e^{-\frac{\theta}{x}}}{(1+\alpha)\left[1-e^{-\frac{\theta}{x}}\right]^3} \exp\left\{-\alpha\left[\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right]\right\}\left[1-\frac{\alpha+\left(1-e^{-\frac{\theta}{x}}\right)}{(1+\alpha)\left(1-e^{-\frac{\theta}{x}}\right)}\exp\left\{-\alpha\left[\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right]\right\}\right]^{\lambda-1} \tag{8}$$

Prior to substitution in (8), the expansion and simplification of the pdf is done as follows:
Let

$$A = \left[1-\frac{\alpha+\left(1-e^{-\frac{\theta}{x}}\right)}{(1+\alpha)\left(1-e^{-\frac{\theta}{x}}\right)}\exp\left\{-\alpha\left[\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right]\right\}\right]^{\lambda-1} \tag{9}$$

Using the generalized binomial theorem on A gives:

$$\left[1-\frac{\alpha+\left(1-e^{-\frac{\theta}{x}}\right)}{(1+\alpha)\left(1-e^{-\frac{\theta}{x}}\right)}\exp\left\{-\alpha\left[\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right]\right\}\right]^{\lambda-1} = \sum_{i=0}^{\infty}(-1)^i\binom{\lambda-1}{i}\frac{\left(\alpha+1-e^{-\frac{\theta}{x}}\right)^i}{(1+\alpha)^i\left(1-e^{-\frac{\theta}{x}}\right)^i}\exp\left\{-\alpha i\left[\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right]\right\} \tag{10}$$

Making use of the result in (10) above, equation (8) becomes:

$$f(x) = \sum_{i=0}^{\infty}\binom{\lambda-1}{i}\frac{(-1)^i\alpha^2\theta\lambda x^{-2}e^{-\frac{\theta}{x}}\left(\alpha+1-e^{-\frac{\theta}{x}}\right)^i}{(1+\alpha)^{i+1}\left(1-e^{-\frac{\theta}{x}}\right)^{i+3}}\exp\left\{-\alpha(i+1)\left[\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right]\right\} \tag{11}$$

First, by expanding the exponential term in (11) using power series, it gives:

$$\exp\left\{-\alpha(i+1)\left[\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right]\right\}=\sum_{k=0}^{\infty}\frac{(-1)^k\left[\alpha(i+1)\right]^k}{k!}\left(\frac{\mathrm{e}^{-\frac{\theta}{x}}}{1-\mathrm{e}^{-\frac{\theta}{x}}}\right)^k \tag{12}$$

Using (12) above and simplifying, (11) becomes

$$f(x)=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}\alpha^2\theta\lambda\left[\alpha(i+1)\right]^k}{(1+\alpha)^{i+1}k!}\binom{\lambda-1}{i}x^{-2}e^{-\frac{\theta}{x}(k+1)}\left(\alpha+1-e^{-\frac{\theta}{x}}\right)^i\left[1-e^{-\frac{\theta}{x}}\right]^{-(i+k+3)} \tag{13}$$

Also, using the generalized binomial theorem, the last term from the above result can be written as:

$$\left[1-\mathrm{e}^{-\frac{\theta}{x}}\right]^{-(i+k+3)}=\sum_{l=0}^{\infty}\frac{\Gamma(l+i+k+3)}{l!\,\Gamma(i+k+3)}\mathrm{e}^{-\frac{\theta}{x}l} \tag{14}$$

Using (14) above in equation (13) and simplifying gives:

$$f(x)=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(l+i+k+3)(-1)^{i+k}\alpha^2\theta\lambda\left[\alpha(i+1)\right]^k}{l!\,\Gamma(i+k+3)(1+\alpha)^{i+1}k!}\binom{\lambda-1}{i}x^{-2}e^{-\frac{\theta}{x}(l+k+1)}\left(\alpha+1-\mathrm{e}^{-\frac{\theta}{x}}\right)^i \tag{15}$$

Again making use of the generalized binomial expansion on the last term from equation (15) above, we have:

$$\left(\alpha+1-\mathrm{e}^{-\frac{\theta}{x}}\right)^i=\sum_{r=0}^{\infty}(-1)^r\binom{i}{r}\mathrm{e}^{-\frac{\theta}{x}r} \tag{16}$$

Hence, the pdf in equation (15) can again be written in its simple form as follows:

$$f(x)=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty}\frac{\Gamma(l+i+k+3)(-1)^{i+k+r}\alpha^2\theta\lambda\left[\alpha(i+1)\right]^k}{l!\,\Gamma(i+k+3)(1+\alpha)^{i+1}k!}\binom{\lambda-1}{i}\binom{i}{r}x^{-2}\mathrm{e}^{-\frac{\theta}{x}(l+k+r+1)} \tag{17}$$

Consequently, the pdf in (17) can also be written in its simplest form as:

$$f(x)=\eta_{i,k,l,r}x^{-2}\mathrm{e}^{-\frac{\theta}{x}(l+k+r+1)} \tag{18}$$

where $\eta_{i,k,l,r}=\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty}\frac{\Gamma(l+i+k+3)(-1)^{i+k+r}\alpha^2\theta\lambda\left[\alpha(i+1)\right]^k}{l!\,\Gamma(i+k+3)(1+\alpha)^{i+1}k!}\binom{\lambda-1}{i}\binom{i}{r}$.

Now, using the linear form of the pdf of the ExpOLinInExD in equation (18), the n^{th} ordinary moment of the ExpOLinInExD is derived as follows:

$$\mu_n^{'} = E\left(X^n\right) = \int_0^{\infty} x^n f(x)dx = \int_0^{\infty} \eta_{i,k,l,r} x^{n-2} \mathrm{e}^{-\frac{\theta}{x}(k+l+r+1)} dx \tag{19}$$

Making use of integration by substitution method in equation (19) leads to the following operations:

Let $u = \frac{\theta}{x}\left(k+l+r+1\right) \Rightarrow x = u^{-1}\theta\left(k+l+r+1\right)$ which implies that $\frac{du}{dx} = -\frac{\theta(k+l+r+1)}{x^2} \Rightarrow dx = -\frac{x^2 du}{\theta(k+l+r+1)}$.

Substituting for $x,\ u$ and dx in equation (19) and simplifying; we have:

$$\mu_n^{'} = E\left(X^n\right) = \eta_{i,k,l,r}\left(\frac{\left(\theta(k+l+r+1)\right)^n}{\theta(k+l+r+1)}\right)\int_0^{\infty} u^{-n} e^{-u} du = \eta_{i,k,l,r}\left(\frac{\left(\theta(k+l+r+1)\right)^n}{\theta(k+l+r+1)}\right)\int_0^{\infty} u^{1-n-1} e^{-u} du \tag{20}$$

Hence, recall that $\int_0^{\infty} t^{m-1} e^{-t} dt = \Gamma(m)$ and that $\int_0^{\infty} t^m e^{-t} dt = \int_0^{\infty} t^{m+1-1} e^{-t} dt = \Gamma(m+1)$

Thus we obtain the n^{th} ordinary moment of X for the ExpOLinInExD as follows:

$$\mu_n^{'} = E\left(X^n\right) = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty}\binom{\lambda-1}{i}\binom{i}{r}\frac{\Gamma\left(l+i+k+3\right)\left(-1\right)^{i+k+r}\alpha^2\theta\lambda\left[\alpha\left(i+1\right)\right]^k\Gamma\left(1-n\right)}{l!\Gamma\left(i+k+3\right)\left(1+\alpha\right)^{i+1}k!\left[\theta\left(k+l+r+1\right)\right]^{1-n}} \tag{21}$$

Using the above moments can help in the calculation of variation, skewness and kurtosis with appropriate formulas.

3.2 Moment Generating Function

The moment generating function of a random variable X can be obtained as

$$M_x(t) = E\left[e^{tx}\right] = \int_{-\infty}^{\infty} e^{tx} f(x)dx \tag{22}$$

Recall that by power series expansion,

$$e^{tx} = \sum_{m=0}^{\infty}\frac{(tx)^m}{m!} = \sum_{r=0}^{\infty}\frac{t^m}{m!}x^m \tag{23}$$

Therefore, the moment generating function can also be expressed as:

$$M_x(t) = \sum_{m=0}^{\infty}\frac{t^m}{m!}\int_{-\infty}^{\infty} x^m f(x)dx = \sum_{m=0}^{\infty}\frac{t^m}{m!}E\left(X^m\right) = \sum_{m=0}^{\infty}\frac{t^m}{m!}\left[\mu_m^{'}\right]$$

Using the result in (23) and simplifying the integral in (22) gives:

$$M_x(t)=\sum_{m=0}^{\infty}\frac{t^m}{m!}\left[\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty}\binom{\lambda-1}{i}\binom{i}{r}\frac{\Gamma(l+i+k+3)(-1)^{i+k+r}\alpha^2\theta\lambda\left[\alpha(i+1)\right]^k\Gamma(1-m)}{l!\Gamma(i+k+3)(1+\alpha)^{i+1}k!\left[\theta(k+l+r+1)\right]^{1-m}}\right] \tag{24}$$

3.3 Characteristics Function

A representation for the characteristics function is given by

$$\phi_x(t)=E\left(e^{itx}\right)=\int_0^{\infty}e^{itx}f(x)\,dx \tag{25}$$

Recall that by power series expansion,

$$\mathrm{e}^{itx}=\sum_{m=0}^{\infty}\frac{(itx)^m}{m!}=\sum_{m=0}^{\infty}\frac{(it)^m}{m!}x^m \tag{26}$$

Hence, simple algebra and use of (26) above produces the following results:

$$\phi_x(t)=\sum_{m=0}^{\infty}\frac{(it)^m}{m!}\int_0^{\infty}x^m f(x)\,dx=\sum_{m=0}^{\infty}\frac{(it)^m}{m!}E\left(X^m\right)=\sum_{m=0}^{\infty}\frac{(it)^m}{m!}\left[\mu_m^{'}\right]$$

$$\phi_x(t)=\sum_{m=0}^{\infty}\frac{(it)^m}{m!}\left[\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{r=0}^{\infty}\binom{\lambda-1}{i}\binom{i}{r}\frac{\Gamma(l+i+k+3)(-1)^{i+k+r}\alpha^2\theta\lambda\left[\alpha(i+1)\right]^k\Gamma(1-m)}{l!\Gamma(i+k+3)(1+\alpha)^{i+1}k!\left[\theta(k+l+r+1)\right]^{1-m}}\right] \tag{27}$$

3.4 Quantile Function

According to [26], the quantile function for any distribution with cdf, *F*(*x*) is defined in the form, $Q(u)=X_q=F^{-1}(u)$, where $Q(u)$ is the quantile function of *F*(*x*) for $0<u<1$

Using the cdf of the ExpOLinInExD and inverting it as above gives the quantile function as follows:

$$F(x)=\left\{1-\frac{\alpha+\left(1-e^{-\frac{\theta}{x}}\right)}{(1+\alpha)\left(1-e^{-\frac{\theta}{x}}\right)}\exp\left\{-\alpha\left[\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right]\right\}\right\}^{\lambda}=u \tag{28}$$

Collecting like terms and simplifying equation (27) above gives:

$$-(\alpha+1)\left(1-u^{\frac{1}{\lambda}}\right)e^{-(\alpha+1)}=-\frac{\alpha+1-\mathrm{e}^{-\frac{\theta}{x}}}{1-\mathrm{e}^{-\frac{\theta}{x}}}\mathrm{e}^{-\frac{\alpha+1-e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}} \tag{29}$$

Considering equation (29), we can see that $-\frac{\alpha+1-e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}$ is the Lambert function of the real argument $-(\alpha+1)(1-u)e^{-(\alpha+1)}$ because the Lambert function is defined as: $w(x)e^{w(x)}=\Re$

It is also important to recall that the Lambert function has two branches with a branching point located at $(-e^{-1},1)$. The lower branch, $W_{-1}(x)$ is defined in the interval $[-e^{-1},1]$ and has a negative singularity for $x\to 0$. The upper branch, $W_0(x)$, is defined for $x\in[-e^{-1},\infty]$. Therefore, (29) can be expressed as:

$$W\left(-(\alpha+1)\left(1-u^{\frac{1}{\lambda}}\right)e^{-(\alpha+1)}\right)=-\frac{\alpha+1-e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}} \tag{30}$$

Recall that for any $\alpha>0$ and $u\in(0,1)$, $\frac{\alpha+1-e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}>1$ and $\left((\alpha+1)(1-u)e^{-(\alpha+1)}\right)<0$. Hence investigating the lower branch of the Lambert function, equation (30) can be stated as:

$$W_{-1}\left(-(\alpha+1)\left(1-u^{\frac{1}{\lambda}}\right)e^{-(\alpha+1)}\right)=-\frac{\alpha+1-e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}} \tag{31}$$

Solving equation (31), the quantile function of the ExpOLinlnExD is expressed as:

$$Q(u)=\left\{-\frac{1}{\theta}\log\left[\alpha\left(1+W_{-1}\left(-(\alpha+1)\left(1-u^{\frac{1}{\lambda}}\right)e^{-(\alpha+1)}\right)\right)^{-1}+1\right]\right\}^{-1} \tag{32}$$

where $W_{-1}(.)$ stands for the negative branch of the Lambert function and *u* is uniform interval (0,1).

From (32), the median of *X* based on the ExpOLinlnExD is calculated by letting *u*=0.5 and this gives:

$$Median=\left\{-\frac{1}{\theta}\log\left[\alpha\left(1+W_{-1}\left(-(\alpha+1)\left(\tfrac{1}{2}\right)^{\frac{1}{\lambda}}e^{-(\alpha+1)}\right)\right)^{-1}+1\right]\right\}^{-1} \tag{33}$$

Consequently, random samples could be obtained from ExpOLinlnExD from (32) by letting $Q(u)=X$ which is known as inverse transformation method of simulation. Hence it gives the representation:

$$X=\left\{-\frac{1}{\theta}\log\left[\alpha\left(1+W_{-1}\left(-(\alpha+1)\left(1-u^{\frac{1}{\lambda}}\right)e^{-(\alpha+1)}\right)\right)^{-1}+1\right]\right\}^{-1} \tag{34}$$

From (32) and according to Kennedy and Keeping [27], the Bowley's measure of skewness is defined as:

$$SK=\frac{Q\left(3/4\right)-2Q\left(1/2\right)+Q\left(1/4\right)}{Q\left(3/4\right)-Q\left(1/4\right)} \tag{35}$$

Similarly, Moors [28] defined Moors' kurtosis based on octiles from (32) as:

$$KT=\frac{Q\left(7/8\right)-Q\left(5/8\right)-Q\left(3/8\right)+\left(1/8\right)}{Q\left(6/8\right)-Q\left(1/8\right)} \tag{36}$$

where $Q(.)$ is calculated from equation (32).

3.5 Reliability Analysis of the ExpOLinInExD

The Survival function describes the likelihood that a system or an individual will not fail after a given time. Mathematically, the survival function is given by:

$$S\left(x\right)=1-F\left(x\right) \tag{37}$$

Applying the cdf of the ExpOLinInExD in (37), the survival function for the ExpOLinInExD is obtained as:

$$S(x)=1-\left\{1-\frac{\alpha+\left(1-e^{-\frac{\theta}{x}}\right)}{\left(1+\alpha\right)\left(1-e^{-\frac{\theta}{x}}\right)}\exp\left\{-\alpha\left[\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right]\right\}\right\}^{\lambda}$$

(38)

Hazard function is a function that describes the chances that a product or component will breakdown over an interval of time. It is mathematically defined as:

$$h\left(x\right)=\frac{f\left(x\right)}{S\left(x\right)}=\frac{f\left(x\right)}{1-F\left(x\right)} \tag{39}$$

Therefore, our definition of the hazard rate of the ExpOLinInExD is given by

$$h(x)=\frac{\alpha^{2}\theta\lambda x^{-2}\mathrm{e}^{-\frac{\theta}{x}}\exp\left\{-\alpha\left[\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right]\right\}\left[1-\frac{\alpha+\left(1-e^{-\frac{\theta}{x}}\right)}{\left(1+\alpha\right)\left(1-e^{-\frac{\theta}{x}}\right)}\exp\left\{-\alpha\left[\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right]\right\}\right]^{\lambda-1}}{\left(1+\alpha\right)\left[1-e^{-\frac{\theta}{x}}\right]^{3}\left\{1-\left[1-\frac{\alpha+\left(1-e^{-\frac{\theta}{x}}\right)}{\left(1+\alpha\right)\left(1-e^{-\frac{\theta}{x}}\right)}\exp\left\{-\alpha\left[\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right]\right\}\right]^{\lambda}\right\}} \tag{40}$$

where $\alpha, \theta, \lambda > 0$.

The survival function (SF) and hazard function (HF) of ExpOLinlnExD based on arbitrary parameter values are presented as followed below in Fig. 2:

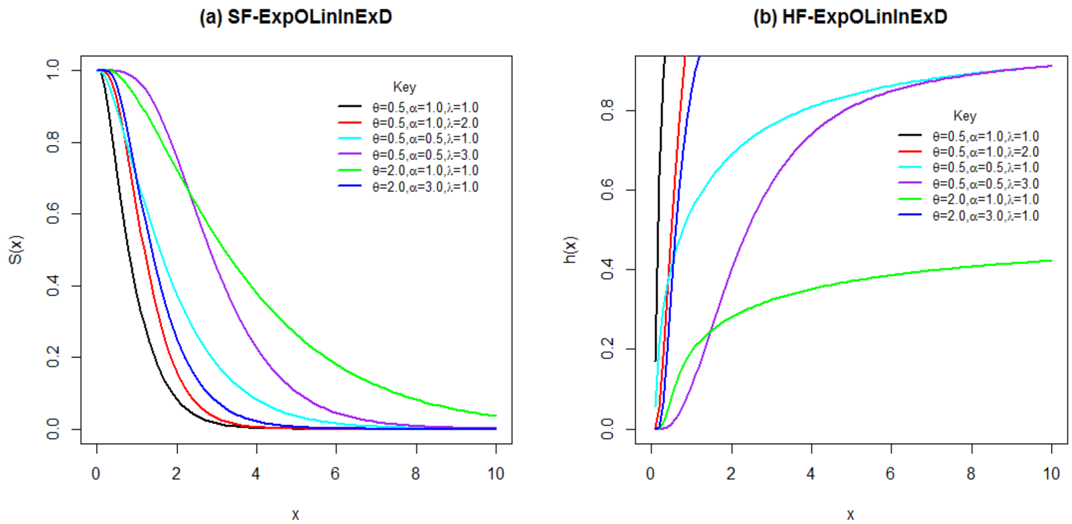

Fig. 2. (a)-SF and (b)-HF of ExpOLinlnExD for selected values of the parameters

Fig. **2(a)** describes the behavior of the survival function, it indicates that the likelihood of survival for any random variable following the proposed distribution is higher at the beginning or initial stage and decreases as time increases and tends to zero at infinity. Fig. 2(b) also shows that the proposed distribution has increasing failure rate which implies that the probability of failure for any random variable following the distribution increases as time increases, that is, probability of death increases as the component ages.

4. ESTIMATION OF UNKNOWN PARAMETERS OF ExpOLinlnExD

Let $X_1, X_2, ..., X_n$ be a sample of size "n" independently and identically distributed random variables from the ExpOLinlnExD with unknown parameters α, θ and λ.

The likelihood function is given by:

$$L(\underline{X} \mid \alpha, \theta, \lambda) = \frac{\left(\alpha^2 \theta \lambda\right)^n \prod_{i=1}^{n}\left(x_i^{-2} \mathrm{e}^{-\frac{\theta}{x_i}}\right)}{(1+\alpha)^n \prod_{i=1}^{n}\left[1-\mathrm{e}^{-\frac{\theta}{x}}\right]^3} \mathrm{e}^{-\alpha \sum_{i=1}^{n}\left[\frac{\mathrm{e}^{-\frac{\theta}{x_i}}}{1-\mathrm{e}^{-\frac{\theta}{x_i}}}\right]} \prod_{i=1}^{n}\left[1-\frac{\alpha+\left(1-\mathrm{e}^{-\frac{\theta}{x_i}}\right)}{(1+\alpha)\left(1-\mathrm{e}^{-\frac{\theta}{x_i}}\right)} \exp\left\{-\alpha\left[\frac{\mathrm{e}^{-\frac{\theta}{x_i}}}{1-\mathrm{e}^{-\frac{\theta}{x_i}}}\right]\right\}\right]^{\lambda-1}$$

Let the log-likelihood function be $l = \log L(\underline{X} \mid \alpha, \theta, \lambda)$ therefore:

$$l = 2n\log\alpha + n\log\theta + n\log\lambda - n\log(1+\alpha) - 2\sum_{i=1}^{n}\log x_i - \theta\sum_{i=1}^{n} x_i^{-1} - 3\sum_{i=1}^{n}\log\left(1-e^{-\frac{\theta}{x}}\right) - \alpha\sum_{i=1}^{n}\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)$$
$$+(\lambda-1)\sum_{i=1}^{n}\log\left\{1-\frac{\alpha+\left(1-\mathrm{e}^{-\frac{\theta}{x_i}}\right)}{(1+\alpha)\left(1-\mathrm{e}^{-\frac{\theta}{x_i}}\right)}\exp\left\{-\alpha\left[\frac{\mathrm{e}^{-\frac{\theta}{x_i}}}{1-\mathrm{e}^{-\frac{\theta}{x_i}}}\right]\right\}\right\} \quad (41)$$

Differentiating l partially with respect to α,θ and λ respectively gives;

$$\frac{\partial l}{\partial \alpha}=\frac{2n}{\alpha}-\frac{n}{(\alpha+1)}-\sum_{i=1}^{n}\left(\frac{e^{-\frac{\theta}{x}}}{1-e^{-\frac{\theta}{x}}}\right)+(\lambda-1)\sum_{i=1}^{n}\left\{\frac{\mathrm{e}^{-\frac{\theta}{x_i}}\exp\left\{-\alpha\left[\frac{\mathrm{e}^{-\frac{\theta}{x_i}}}{1-\mathrm{e}^{-\frac{\theta}{x_i}}}\right]\right\}\left[1-\frac{\left(\alpha+1-\mathrm{e}^{-\frac{\theta}{x_i}}\right)\left(\alpha+1-(\alpha+1)\mathrm{e}^{-\frac{\theta}{x_i}}\right)}{\left[1-\mathrm{e}^{-\frac{\theta}{x_i}}\right]}\right]}{\left[\alpha+1-(\alpha+1)\mathrm{e}^{-\frac{\theta}{x_i}}\right]^2\left[1-\frac{\alpha+\left(1-\mathrm{e}^{-\frac{\theta}{x_i}}\right)}{(1+\alpha)\left(1-\mathrm{e}^{-\frac{\theta}{x_i}}\right)}\exp\left\{-\alpha\left[\frac{\mathrm{e}^{-\frac{\theta}{x_i}}}{1-\mathrm{e}^{-\frac{\theta}{x_i}}}\right]\right\}\right]}\right\} \tag{42}$$

$$\frac{\partial l}{\partial \theta}=\frac{n}{\theta}-\sum_{i=1}^{n}x^{-1}{}_i+3\sum_{i=1}^{n}\left\{\frac{x_i^{-1}\mathrm{e}^{-\frac{\theta}{x_i}}}{\left(1-\mathrm{e}^{-\frac{\theta}{x_i}}\right)}\right\}+\alpha\sum_{i=1}^{n}\left\{\frac{x_i^{-1}\mathrm{e}^{-\frac{\theta}{x_i}}}{\left[1-\mathrm{e}^{-\frac{\theta}{x_i}}\right]^2}\right\}+\alpha(\lambda-1)\sum_{i=1}^{n}\left\{\frac{\mathrm{e}^{-\frac{\theta}{x_i}}\exp\left\{-\alpha\left[\frac{\mathrm{e}^{-\frac{\theta}{x_i}}}{1-\mathrm{e}^{-\frac{\theta}{x_i}}}\right]\right\}\left[(\alpha+1)-\frac{\left(\alpha+1-\mathrm{e}^{-\frac{\theta}{x_i}}\right)\left(\alpha+1-(\alpha+1)\mathrm{e}^{-\frac{\theta}{x_i}}\right)}{\left[1-\mathrm{e}^{-\frac{\theta}{x_i}}\right]^2}\right]}{x_i\left[\alpha+1-(\alpha+1)\mathrm{e}^{-\frac{\theta}{x_i}}\right]^2\left[1-\frac{\alpha+\left(1-\mathrm{e}^{-\frac{\theta}{x_i}}\right)}{(1+\alpha)\left(1-\mathrm{e}^{-\frac{\theta}{x_i}}\right)}\exp\left\{-\alpha\left[\frac{\mathrm{e}^{-\frac{\theta}{x_i}}}{1-\mathrm{e}^{-\frac{\theta}{x_i}}}\right]\right\}\right]}\right\} \tag{43}$$

$$\frac{\partial l}{\partial \lambda}=\frac{n}{\lambda}+\sum_{i=1}^{n}\log\left[1-\frac{\alpha+\left(1-\mathrm{e}^{-\frac{\theta}{x_i}}\right)}{(1+\alpha)\left(1-\mathrm{e}^{-\frac{\theta}{x_i}}\right)}\exp\left\{-\alpha\left[\frac{\mathrm{e}^{-\frac{\theta}{x_i}}}{1-\mathrm{e}^{-\frac{\theta}{x_i}}}\right]\right\}\right] \tag{44}$$

To solve for the maximum likelihood estimates, equation (42), (43) and (44) are set to zero (0) and the solution of the non-linear system of equations is obtained to give the maximum likelihood estimates (MLEs) of parameters $\hat{\alpha},\hat{\theta}$ and $\hat{\lambda}$. For this paper, the "AdequacyModel" package in R software was used to obtain the estimates of the parameters of the proposed distribution using real life data sets since the manual solution cannot be obtained.

5. APPLICATIONS

This particular section of the paper considered the application of the proposed distribution to two real life datasets to validate modeling ability of the proposed distribution compared to some other existing extensions of the inverse exponential distribution. Some of the distributions compared under this section are as follows: the Exponentiated Exponential Inverse Exponential distribution (ExpExInExD), (the proposed model), the Exponential Inverse Exponential distribution (ExInExD), Odd Lindley Inverse Exponential distribution (OLINExD), Lindley distribution (LIND), Inverse Exponential distribution (InExD) and Exponential distribution (ExD). To select the most fitted distribution to each of the two datasets, the following model selection information criteria were used. They are the value of the log-likelihood function evaluated at the MLEs (ℓ), Akaike Information Criterion, *AIC,* Consistent Akaike Information Criterion, *CAIC*, Bayesian Information Criterion, *BIC*, Hannan Quin Information Criterion, *HQIC*, Anderson-Darling (A*), Cramèr-Von Mises (W*) and Kolmogorov-smirnov (K-S) statistics. More about these statistics A*, W* and K-S can be seen in [29]. Some of these statistics are computed using the following formulas:

$AIC=-2\ell+2k$, $BIC=-2\ell+k\log(n)$, $CAIC=-2\ell+\frac{2kn}{(n-k-1)}$ and $HQIC=-2\ell+2k\log\left[\log(n)\right]$

Where ℓ denotes the value of log-likelihood function evaluated at the *MLEs*, *k* is the number of model parameters and *n* is the sample size. Decisively, the distribution with the lowest values of these criteria is considered to be the most fitted model to the dataset. Also, all the required computations are performed using the R package "AdequacyModel".

5.1 Application to Infant Mortality Rate (IMR)

This section presents a dataset on infant mortality rate in Nigeria from the year 1964 to the year 2019 with its descriptive statistics or summary.

This infant mortality rate per 1,000 of population in Nigeria from 1964 to 2019 is as given below.

192.71657, 188.26883, 183.91102, 179.70244, 175.60699, 171.38654, 166.80645, 162.17431, 157.23233, 152.27853, 147.39343, 142.71809, 138.29272, 134.21357, 130.67731, 127.81633, 125.55941, 123.88542, 122.88514, 122.49766, 122.52200, 122.85743, 123.36316, 123.86249, 124.21932, 124.40590, 124.31438, 124.04075, 123.65079, 123.14077, 122.40752, 121.33929, 119.82728, 117.87892, 115.53888, 112.84664, 109.97357, 107.02932, 104.03164, 101.04458, 98.10692, 95.21899, 92.53070, 90.07801, 87.96513, 86.11200, 84.57112, 83.25569, 82.19877, 81.23519, 80.41503, 79.51666, 78.52203, 77.27902, 75.74067, 74.16032.

Data source: www.data.unicef.org

The Table 1 shows a good summary of the above dataset with some important explanations:

The descriptive statistics in Table 1 shows that the infant mortality rate skewed to the right with a very low kurtosis not far different from that of the normal distribution.

The results from this R package and the commands are shown in tables as follows: Table 2 lists the Maximum Likelihood Estimates of the model parameters, Table 3 presents the statistics AIC, CAIC, BIC and HQIC while A*, W* and K-S for the fitted models are given in Table 4.

The Fig. 3 presents a histogram and estimated densities and cdfs of the fitted models to the dataset.

The results from Tables 3 and .4 show that the proposed distribution (ExpOLinInExD) fits the infant mortality rate data better compared to the other five fitted distributions (OLINExD, ExInExD, LinD, InExD and ExD) based on the information criteria (AIC, CAIC, BIC and HQIC). Also, the statistics in Table 4 reveal that the proposed model fits the dataset than the other distributions because the ExpOLinInExD has the minimum values of A^*, W^* and K-S statistic compared to every other model fitted to the dataset.

The histogram of the dataset together with the fitted densities and estimated cumulative distribution functions given in Fig. 3 also confirm that the proposed model analyses the dataset better than the LIND, OLINExD, ExInExD,InExD and the conventional ExD. Also, the probability plots presented in Fig. 4 prove that the proposed distribution (ExpOLinInExD) is more flexible than the other five distributions (LIND, OLINExD, ExInExD, ExD and InExD) as already revealed previously in Tables 3 and 4 as well as Fig. 3 respectively.

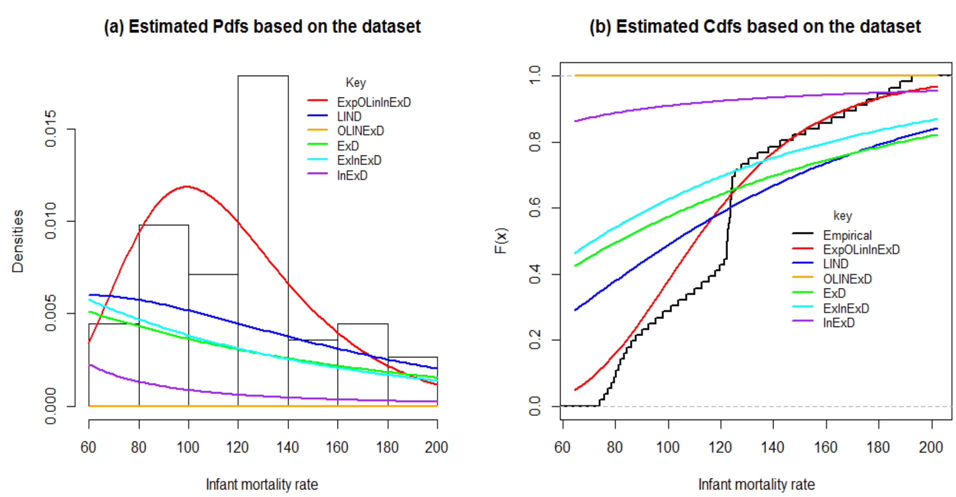

Fig. 3. Histogram and plots of the estimated densities and Cdfs of the fitted distributions to the dataset

Table 1. Descriptive statistics for the dataset

parameters	n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
Values	56	74.16	94.55	122.69	131.56	120.74	192.72	983.73	0.5022	-0.3814

Table 2. Maximum likelihood parameter estimates for the dataset

Distribution	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$
ExpOLinInExD	3.4845787	0.1336563	9.1914488
LIND	-	0.01626093	-
OLINExD	4.503531	1.303863	-
ExD	0.00853896	-	-
ExInExD	8.14101338	0.08348915	-
InExD	9.545694	-	-

Table 3. The statistics ℓ, AIC, CAIC, BIC and HQIC based on the dataset used

Distribution	$\hat{\ell}$	AIC	CAIC	BIC	HQIC	Ranks
ExpOLinInExD	273.4536	552.9072	553.3687	558.9832	555.2629	1st
LIND	305.0766	612.1532	612.2273	614.1785	612.9384	2nd
OLINExD	1839.102	3682.204	3682.43	3686.254	3683.774	3rd
ExD	324.4683	650.9366	651.0107	652.962	651.7219	4th
ExInExD	323.5324	651.0648	651.2912	655.1155	652.6353	5th
InExD	411.5954	825.1908	825.2649	827.2162	825.9761	6th

Table 4. The A^*, W^*, K-S statistic and P-values based on the dataset used

Distribution	A^*	W^*	K-S	P-Value (K-S)	Ranks
ExpOLinInExD	1.040266	0.1914481	0.19037	0.02988	1st
LIND	1.003272	0.1781881	0.34529	1.814e-06	2nd
OLINExD	2.71767	0.3955851	1.0000	7.772e-16	3rd
ExD	1.002866	0.1782446	0.46914	7.992e-12	4th
ExInExD	1.001909	0.1783947	0.51303	3.386e-14	5th
InExD	1.078119	0.201275	0.87922	7.772e-16	6th

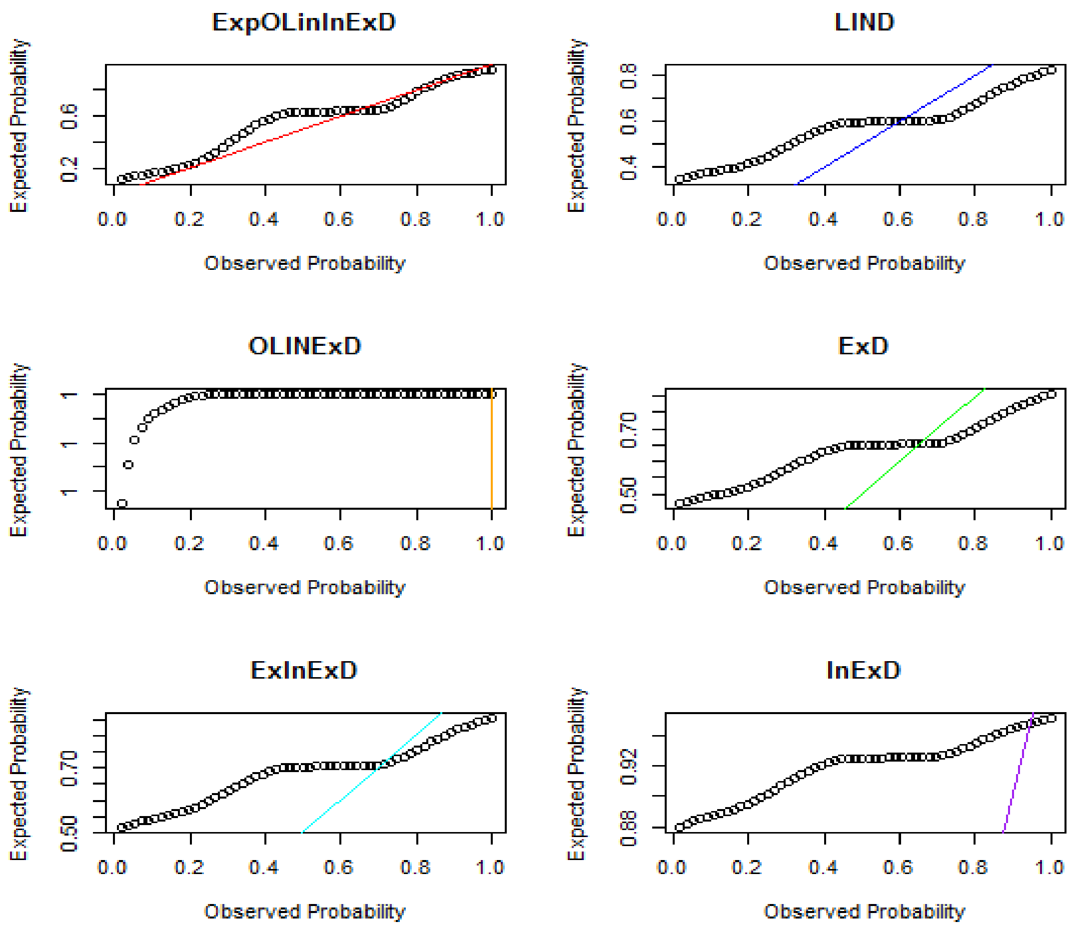

Fig. 4. Probability plots for the six fitted distributions based on the infant mortality rate dataset

The above discussions have proven the general statement that adding parameter(s) to continuous probability distributions produce distributions with greater flexibility in modeling real life data as it has already been reported by many other authors in the previous studies.

5.2 Application to Mother-to-Child HIV Transmission Rate (MTCHIVTR)

This section presents a dataset on the rate of mother-to-child transmission of HIV (Human Immunodeficiency Virus) in Nigeria from the year 2000 to the year 2019. The descriptive statistics of the dataset is also presented.

The mother-to-child HIV transmission rate per 1,000 of population in Nigeria between 2000 and 2019 is as given below.

37.35, 37.08, 37.00, 36.98, 36.79, 36.75, 34.35, 32.96, 31.84, 30.35, 30.53, 28.96, 26.71, 22.50, 19.84, 20.04, 19.44, 20.82, 22.09, 22.16

Data source: www.data.unicef.org

The Table 5 presents a summary of the above dataset with some important details:

Following the descriptive statistics in Table 5, it is clear that the rate of transmission of HIV from mother to child is bimodal and approximately normally distributed.

The Fig. 5 shows the trend in the rate of mother-to-child HIV transmission from 2000 to 2019 using a bar chart.

Table 5. Descriptive statistics for the dataset

Parameters	n	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
Values	20	19.44	22.14	30.44	36.76	29.23	37.35	47.55	-0.18919	-1.55278

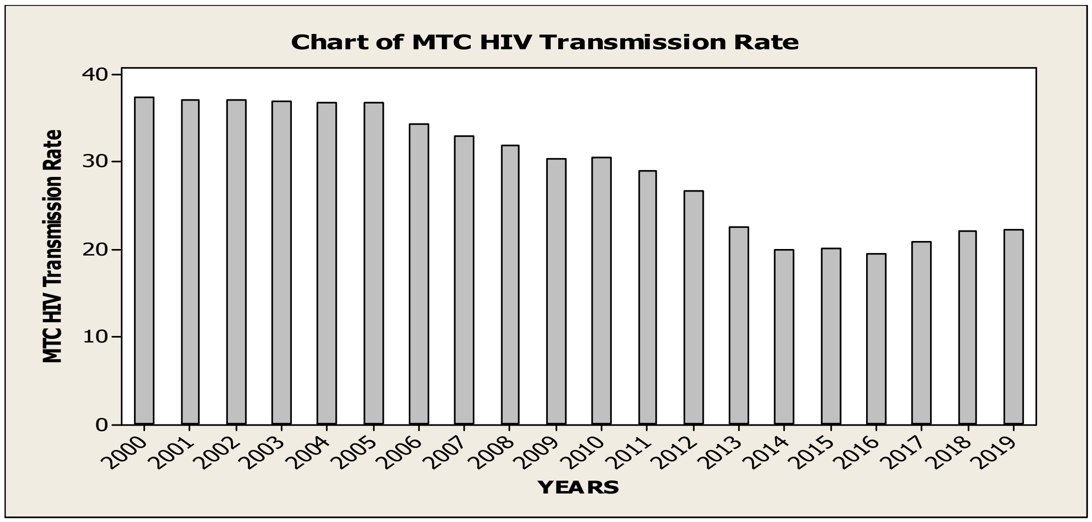

Fig. 5. A Bar chart showing the trend of mother-to-child HIV transmission rate in Nigeria from 2000 to 2019

Following the spread of the dataset with the bar chart in Fig. 5, it can be said that mother-to-child HIV transmission was at its high rate as from the year 2000 to 2005 with a non-decreasing rate. The transmission rate experienced a decreasing trend as from the year 2006 to 2014 but what we have from the year 2015 to 2019 is certainly an increasing pattern in the rate of mother-to-child transmission of HIV which indicates that more efforts or research need to be put in place to adequately reduce or eradicate the increasing rate of mother-to-child HIV transmission in Nigeria.

Applications of the selected models to this data has been done and the results are presented as follows: Table 6 lists the Maximum Likelihood Estimates of the model parameters, Table 7 presents the statistics AIC, CAIC, BIC and HQIC while A*, W* and K-S for the fitted models are given in Table 8.

The Fig. 6 presents a histogram and estimated densities and cdfs of the fitted models to the dataset.

Based on the results from Table 7, it is revealed that the proposed distribution (ExpOLinInExD) fits the MTCHIVTR data better as compared to the other five fitted distributions (OLINExD, ExInExD, LinD, InExD and ExD) using the information criteria (AIC, CAIC, BIC and HQIC). This is also evident from the statistics in Table 8 which show that the proposed model fits the dataset better than the other fitted distributions, because the ExpOLinInExD has the minimum values of A^*, W^* and K-S compared to the other fitted models.

Also, the histogram of the dataset together with the fitted densities and estimated cumulative distribution functions in Fig. 6 also confirm that the proposed model analyses the dataset better than the LIND, OLINExD, ExInExD,InExD and the conventional ExD. Also, the probability plots presented in Fig. 6 show that the proposed distribution (ExpOLinInExD) is more flexible than the other five distributions (LIND, OLINExD, ExInExD, ExD and InExD) as already revealed previously in Tables 7 and 8 as well as Fig. 6.

Table 6. Maximum likelihood parameter estimates for the dataset

Distribution	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\lambda}$
ExpOLinInExD	6.5769480	0.9830842	9.6667805
LIND	-	0.06968643	-
OLINExD	2.2713912	0.1572961	-
ExD	0.03577774	-	-
ExInExD	9.8619772	0.4947511	-
InExD	3.568535	-	-

Table 7. The statistics ℓ, AIC, CAIC, BIC and HQIC based on the dataset used

Distribution	$\hat{\ell}$	AIC	CAIC	BIC	HQIC	Ranks
ExpOLinInExD	68.39668	142.7934	144.2934	145.7806	143.3765	1st
LIND	80.9853	163.9706	164.1928	164.9663	164.165	2nd
OLINExD	80.91053	165.8211	166.5269	167.8125	166.2098	3rd
ExD	87.52211	177.0442	177.2664	178.0399	177.2386	4th
ExInExD	84.74545	173.4909	174.1968	175.4824	173.8797	5th
InExD	111.0148	224.0297	224.2519	225.0254	224.2241	6th

Table 8. The A^*, W^*, K-S statistic and P-values based on the dataset used

Distribution	A^*	W^*	K-S	P-Value (K-S)	Ranks
ExpOLinInExD	1.093851	0.1666872	0.19085	0.4087	1st
LIND	1.042332	0.1534715	0.4152	0.001231	2nd
OLINExD	1.04276	0.1536098	0.41068	0.00145	3rd
ExD	1.044428	0.1540887	0.50118	3.587e-05	4th
ExInExD	1.052848	0.1565771	0.52702	1.051e-05	5th
InExD	1.111589	0.1715118	0.8323	6.661e-16	6th

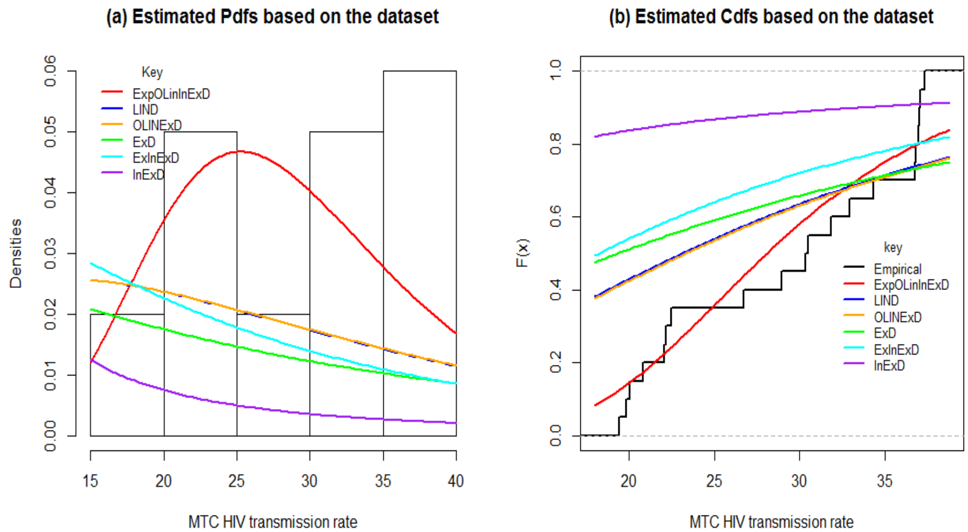

Fig. 6. Histogram and plots of the estimated densities and cdfs of the fitted distributions to the dataset

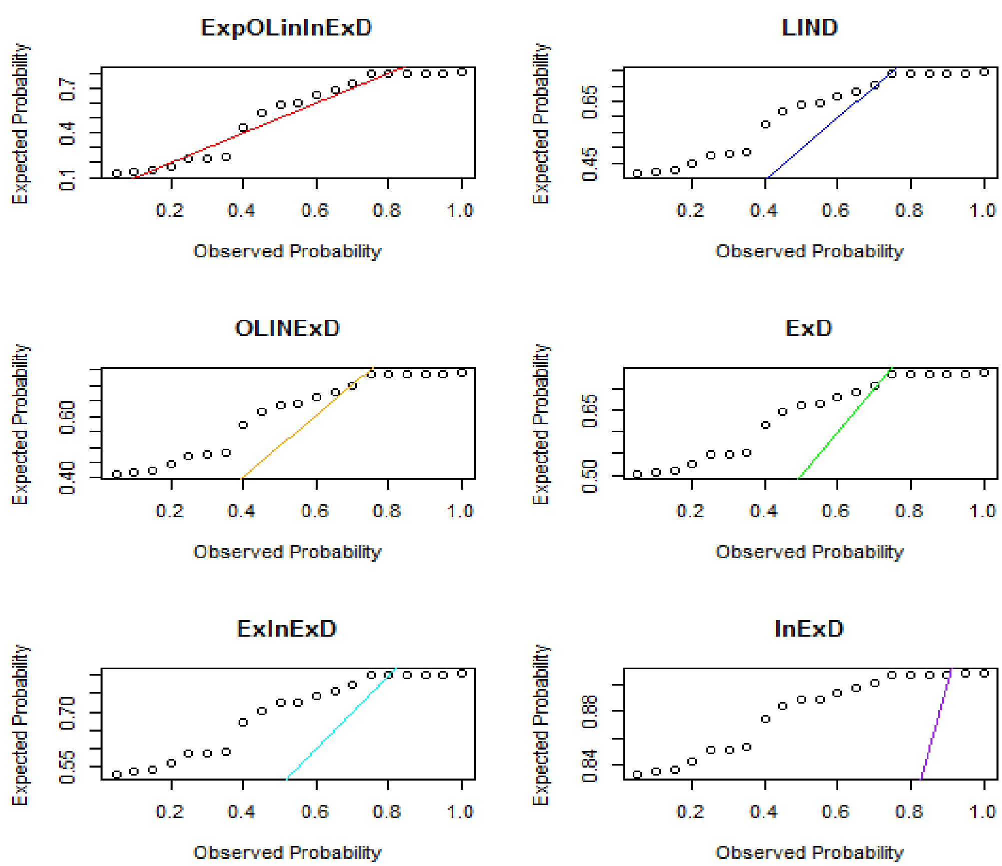

Fig. 7. Probability plots for the six fitted distributions based on the MTCHIVTR dataset

Again, this analysis has proven the general statement that adding parameter(s) to continuous probability distributions produce distributions with greater flexibility in modeling real life data as it has already been reported by many other authors in the previous studies.

6. CONCLUSION

This article presents a new continuous distribution called "exponentiated odd Lindley inverse exponential distribution". The article has generated some important properties of the new distribution such as the moment, moment generating function, characteristics function, quantile function, coefficient of skewness and kurtosis, survival function and hazard function. The unknown parameters of the new distribution have been estimated in the article using the method of maximum likelihood estimation. The proposed distribution has been applied to a dataset on infant mortality rate and mother-to-child HIV transmission rate in Nigeria in comparison with other existing distributions. Data exploratory analysis of the two datasets indicate that both infant mortality and mother-to-child HIV transmission rates are serious health problems in Nigeria and need to be attended to by relevant health agencies. The results from the fitted models based on the IMR and MTCHIVTR datasets show that the exponentiated odd Lindlley inverse exponential distribution fits both datasets much better than the other five fitted distributions. This excellent performance by the proposed model also indicates that it can be a good model especially in the area of survival analysis. This shows that the new model is more flexible than the other five models considered in this study and should be used for modeling other real life situations most especially in health related cases.

CONSENT

It is not applicable.

ETHICAL APPROVAL

It is not applicable.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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